Main Result

Suppose that $\xi : I \twoheadrightarrow J$ (folding map) is a given.

Theorem (Main theorem)

For $\lambda \in \tilde{P}_+$ or $\lambda = \infty$, there exists a unique map $S_\lambda : B_J(\lambda) \to B_I(\lambda)$ such that

$$\mathrm{wt}(S_\lambda(b)) = \mathrm{wt}(b), \quad S_\lambda(\tilde{x}_j b) = \prod_{i \in \xi^{-1}(j)} \tilde{x}_i^{m_i} S_\lambda(b)$$

for $x \in \{e, f\}$, $j \in J$ and $b \in B_J(\lambda)$.

Contents

1. Background: Kashiwara embedding
2. Main result: similarity and folding crystals
3. Fixed point subset
4. Virtual crystals
5. Application: non-simply-laced KR crystals

Contents

1. Background: Kashiwara embedding
2. Main result: similarity and folding crystals
3. Fixed point subset
4. Virtual crystals
5. Application: non-simply-laced KR crystals

Notations

· I : an index set

· $A = (a_{i,j})_{i \in I}$: generalized Cartan matrix

$\rightsquigarrow$ $\mathfrak{g} = \mathfrak{g}(A)$: Kac-Moody algebra associated with A

· P (resp. $P^\vee$) : the weight (resp. coweight) lattice for $\mathfrak{g}$

· P_+ : the set of dominant weights

· ϖ_i : the i-th fundamental weight ($i \in I$)

· $\{\alpha_i \,|\, i \in I\}$: the set of simple roots

· $\{h_i \,|\, i \in I\}$: the set of simple coroots

Elementary crystal

· elementary crystal $\quad \mathbb{B}_i = \{(n)_i \,|\, n \in \mathbb{Z}\}\ (i \in I)$

$$\mathbb{B}_i \quad \cdots \xrightarrow{i} (2)_i \xrightarrow{i} (1)_i \xrightarrow{i} (0)_i \xrightarrow{i} (-1)_i \xrightarrow{i} \cdots$$

$$\mathrm{wt}((m)_i) = m\alpha_i,\ \varepsilon_i((m)_i) = -m,\ \varphi_i((m)_i) = m$$

· $\mathbb{Z}^\infty := \{\underline{x} = (\ldots, x_2, x_1) \,|\, x_k \in \mathbb{Z},\ x_k = 0 \text{ for } k \gg 0\}$

· When a sequence $\underline{i} = (\ldots, i_2, i_1)$ of indices is given, we identify

$$(\ldots, n_2, n_1) \in \mathbb{Z}^\infty \ \longleftrightarrow\ \cdots \otimes (-n_2)_{i_2} \otimes (-n_1)_{i_1} \in \mathbb{B}_{\underline{i}},$$

where $\mathbb{B}_{\underline{i}} = \cdots \otimes \mathbb{B}_{i_2} \otimes \mathbb{B}_{i_1}$.

Crystal structure on $\mathbb{Z}^\infty$

Assume a given sequence $\underline{i} = (i_k)_{k \geq 1}$ satisfies the following properties:

(1) $i_k \neq i_{k+1}$ for all $k \geq 1$,

(2) $|\{k \geq 1 \,|\, i_k = i\}| = \infty$ for all $i \in I$.

For $\underline{x} \in \mathbb{Z}^\infty$, define

- $\sigma_k(\underline{x}) = x_k + \sum_{t=k+1}^{\infty} \langle h_{i_k}, \alpha_{i_t} \rangle x_t$ for $k \geq 1$
- $\sigma^{(i)}(\underline{x}) = \max\{\sigma_k(\underline{x}) \,|\, i_k = i\}$ for $i \in I$
- $M^{(i)}(\underline{x}) = \{k \geq 1 \,|\, \sigma_k(\underline{x}) = \sigma^{(i)}(\underline{x})\}$ for $i \in I$

· $\mathbb{Z}^{\infty}_{\underline{i}}$: the crystal $\mathbb{Z}^{\infty}$ endowed with followings: for $i \in I, \underline{x} \in \mathbb{Z}^{\infty}$,

- $\mathrm{wt}(\underline{x}) = -\sum_{k=1}^{\infty} x_k \alpha_{i_k}$,
- $\varepsilon_i(\underline{x}) = \sigma^{(i)}(\mathbf{x})$, $\varphi_i(\underline{x}) = \varepsilon_i(\underline{x}) + \langle h_i, \mathrm{wt}(\underline{x}) \rangle$
- $\tilde{e}_i \underline{x} = (x'_k)_{k \geq 1}$ where $x'_k = \begin{cases} x_k - \delta_{k, \max M^{(i)}(\mathbf{x})} & \text{if } \sigma^{(i)}(\mathbf{x}) > 0 \\ 0 & \text{otherwise} \end{cases}$
- $\tilde{f}_i \underline{x} = (x''_k)_{k \geq 1}$ where $x''_k = x_k + \delta_{k, \min M^{(i)}(\mathbf{x})}$

Kashiwara embedding

Theorem (Kashiwara 1993)

There exists a unique embedding of crystals

$$\Psi_{\underline{i}} : B(\infty) \hookrightarrow \mathbb{Z}_{\underline{i}}^{\infty} (= \cdots \otimes \mathbb{B}_{i_2} \otimes \mathbb{B}_{i_1})$$

sending $1 \in B(\infty)$ *to the zero vector* $(\ldots, 0, 0) \in \mathbb{Z}_{\underline{i}}^{\infty}$.

Note that

$$\Psi_{\underline{i}}(b) = (\ldots, m_2, m_1)$$

where $m_1 = \varepsilon_{i_1}^*(b)$, and $m_k = \varepsilon_{i_k}^*((\tilde{e}_{i_{k-1}}^*)^{m_{k-1}} \ldots (\tilde{e}_{i_1}^*)^{m_1}(b))$ for $k \geq 2$.

Remark (The image of the embedding $\Psi_{\underline{i}}$)

Theorem (Nakashima-Zelevinsky, 1997)

When $\underline{i}$ satisfies some conditions (the positivity assumption),

$$\begin{aligned}\mathrm{im}\Psi_{\underline{i}}(B(\infty)) \quad = \quad & \{\underline{x}=(x_k)_{k\geq 1}\in\mathbb{Z}^{\infty} \mid x_k\geq 0,\\ & f(\underline{x})\geq 0 \text{ for some linear functions } f\in\Xi_{\underline{i}}.\}\end{aligned}$$

Example

When A is of type A_2 and $\underline{i}=(\ldots,2,1,2,1,2,1)$, we have

$$\mathrm{im}\Psi_{\underline{i}}(B(\infty))=\{\underline{x}=(x_k)_{k\geq 1}\in\mathbb{Z}_{\geq 0}^{\infty} \mid x_2\geq x_3, x_k=0 \text{ for all } k\geq 4\}$$

$$(\ \Xi_{\underline{i}}=\{x_2-x_3\}\cup\{-x_k \mid k\geq 4\}\ )$$

Contents

1. Background: Kashiwara embedding
2. Main result: similarity and folding crystals
3. Fixed point subset
4. Virtual crystals
5. Application: non-simply-laced KR crystals

Setting

A surjective map $\xi : I \twoheadrightarrow J$ is given, and assign $m_i \in \mathbb{Z}_{>0}$ to each $i \in I$.

· $\tilde{\alpha}_j := \sum_{i \in \xi^{-1}(j)} m_i \alpha_i$ for $j \in J$

· $\tilde{P} := \left\{ \lambda \in P \;\middle|\; \begin{array}{l} \text{for any } j \in J,\ \frac{1}{m_i}\langle h_i, \lambda \rangle \text{ is an integer and} \\ \quad \text{does not depend on the choice of } i \in \xi^{-1}(j). \end{array} \right\}$

· $\tilde{h}_j \in \tilde{P}^*$ $(j \in J)$: $\langle \tilde{h}_j, \lambda \rangle := \frac{1}{m_i}\langle h_i, \lambda \rangle$ for any $i \in \xi^{-1}(j), \lambda \in \tilde{P}$

Assumption $\langle h_i, \alpha_j \rangle = 0$ for $i \neq j$ with $\xi(i) = \xi(j)$, $\tilde{\alpha}_j \in \tilde{P}$ for all $j \in J$

$\rightsquigarrow (J, \tilde{P}, \{\tilde{\alpha}_j\}, \tilde{P}^*, \{\tilde{h}_j\})$: Cartan data

· $\tilde{P}_+ := P_+ \cap \tilde{P}$

Main Theorem; similarity

· For $x \in \{e, f\}$ and $j \in J$, write

$$\tilde{X}_j = \prod_{i \in \xi^{-1}(j)} \tilde{x}_i^{m_i}.$$

Note we have $\tilde{x}_i \tilde{x}_j = \tilde{x}_j \tilde{x}_i$ for $i \neq j$ with $\xi(i) = \xi(j)$. ($\Rightarrow \tilde{X}_j$: well-defined)

Theorem (Kashiwara, 1996)

For $\lambda \in \tilde{P}_+$ or $\lambda = \infty$, there exists a unique (injective) map $S_\lambda : (B_J(\lambda), \hat{x}_j) \to (B_I(\lambda), \tilde{x}_i)$ such that

$$\mathrm{wt}(S_\lambda(b)) = \mathrm{wt}(b), \quad S_\lambda(\hat{x}_j b) = \tilde{X}_j S_\lambda(b) \qquad (*)$$

for $x \in \{e, f\}$, $j \in J$ and $b \in B_J(\lambda)$.

Proof of Main Theorem

Notation

· $\mathbb{B}_k^X$: k-th elementary X-crystal ($X = I$ or J, $k \in X$)

· $\widetilde{\mathbb{B}}_j^I := \bigotimes_{i \in \xi^{-1}(j)} (\mathbb{B}_i^I)^{\otimes m_i}$ for $j \in J$

· $\widetilde{\mathbb{B}}_{\underline{j}}^I := \cdots \otimes \widetilde{\mathbb{B}}_{j_2}^I \otimes \widetilde{\mathbb{B}}_{j_1}^I$, $\mathbb{B}_{\underline{j}}^J := \cdots \otimes \mathbb{B}_{j_2}^J \otimes \mathbb{B}_{j_1}^J$ for $\underline{j} = (\ldots, j_2, j_1) \in J^\infty$

Upshot

$$\begin{array}{ccc} B_J(\lambda) & \xrightarrow{\Phi_\lambda^J} & B_J(\infty) \otimes T_\lambda \\ {\scriptstyle S_\lambda}\downarrow & \circlearrowleft & \downarrow{\scriptstyle S_\infty \otimes \mathrm{id}_\lambda} \\ B_I(\lambda) & \xrightarrow[\Phi_\lambda^I]{} & B_I(\infty) \otimes T_\lambda \end{array} \qquad \begin{array}{ccc} B_J(\infty) & \xrightarrow{\Psi^J} & \mathbb{B}_{\underline{j}}^J \\ {\scriptstyle S_\infty}\downarrow & \circlearrowleft & \downarrow{\scriptstyle S} \\ B_I(\infty) & \xrightarrow[\Psi^I]{} & \widetilde{\mathbb{B}}_{\underline{j}}^I \end{array}$$

Sketch of proof

1. Define a map $S : \mathbb{B}^J_{\underline{j}} \to \widetilde{\mathbb{B}}^I_{\underline{j}}$ by

$$S = \cdots \otimes S_{j_2} \otimes S_{j_1} \text{ where } S_j : \mathbb{B}^J_j \to \widetilde{\mathbb{B}}^I_j, S_j(n) = (m_i \cdot n)_{i \in \xi^{-1}(j)},$$

   and show that S is the unique map satisfying $(*)$ with $S(\underline{0}) = \underline{0}$.

2. Define a map $S_\infty : B_J(\infty) \to B_I(\infty)$ by

$$\Psi^I(S_\infty(b)) = (S \circ \Psi^J)(b),$$

   and show that S_∞ is well-defined and that S_∞ satisfies $(*)$.

3. Extend the argument for S_∞ to S_λ using the embedding $\Phi_\lambda : B(\lambda) \hookrightarrow B(\infty) \otimes T_\lambda$.

m-fold scaling

Fix $m \in \mathbb{Z}_{\geq 2}$, and take $I = J, \xi = \mathrm{id}$, and $m_i = m$ for all $i \in I$.

$$\tilde{\alpha}_i = m\alpha_i, \quad \tilde{h}_i = \frac{1}{m} h_i, \quad \tilde{P} = mP$$

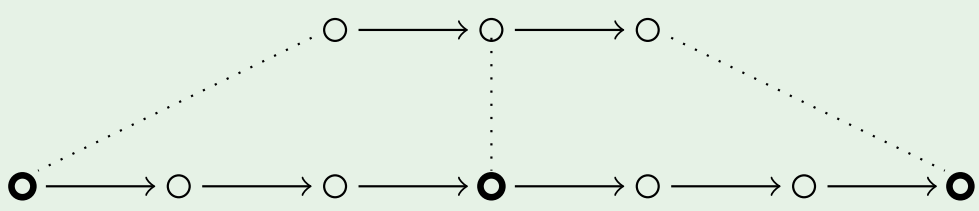
$$\rightsquigarrow \quad \mathrm{wt}(S(b)) = m \cdot \mathrm{wt}(b),\ \varepsilon_i(S(b)) = m \cdot \varepsilon_i(b),\ \varphi_i(S(b)) = m \cdot \varphi_i(b),$$
$$S(\tilde{e}_i b) = \tilde{e}_i^m S(b),\ S(\tilde{f}_i b) = \tilde{f}_i^m S(b)$$

Example ($\mathfrak{sl}_2$ case)

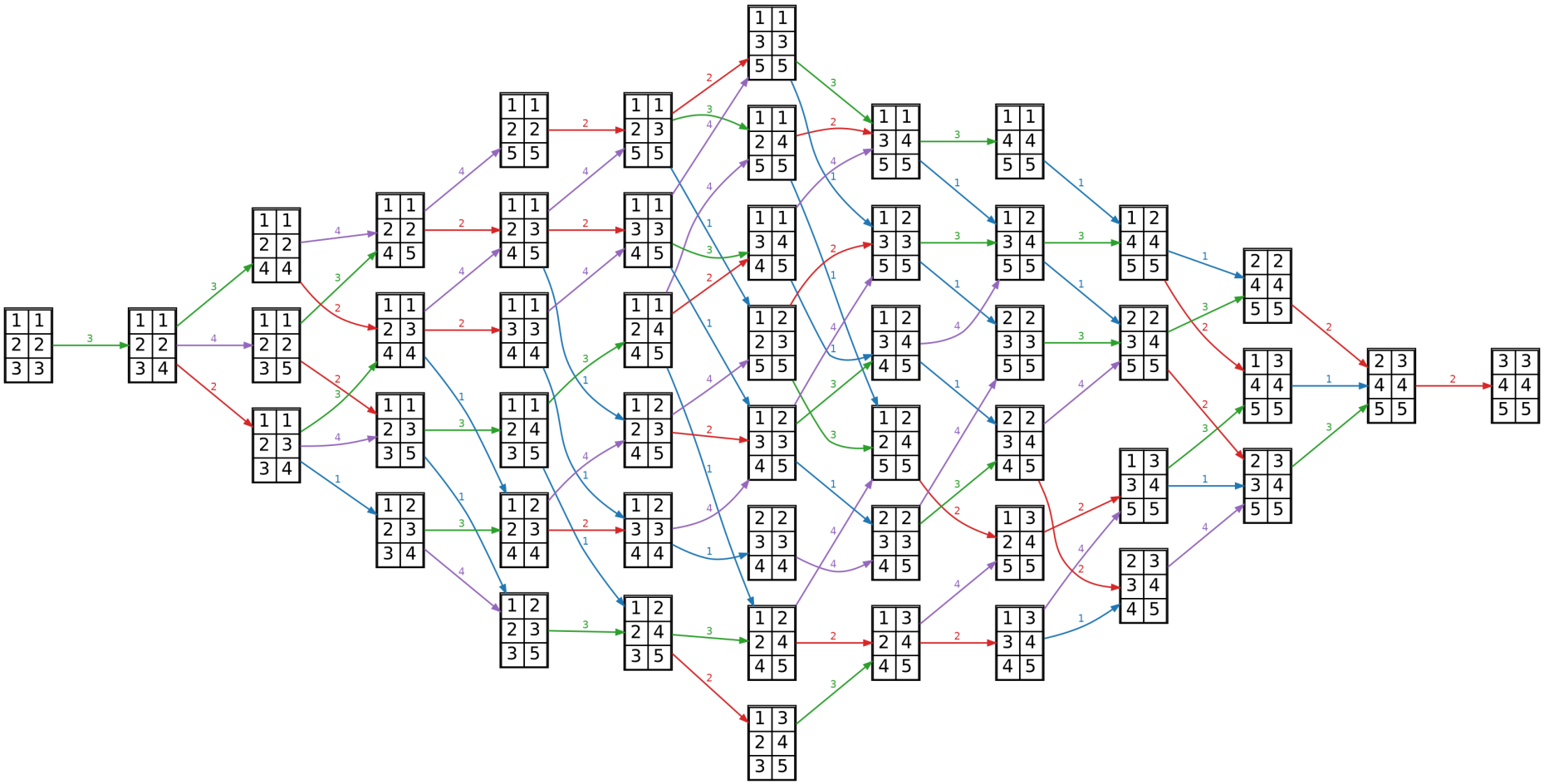
$V(2)$

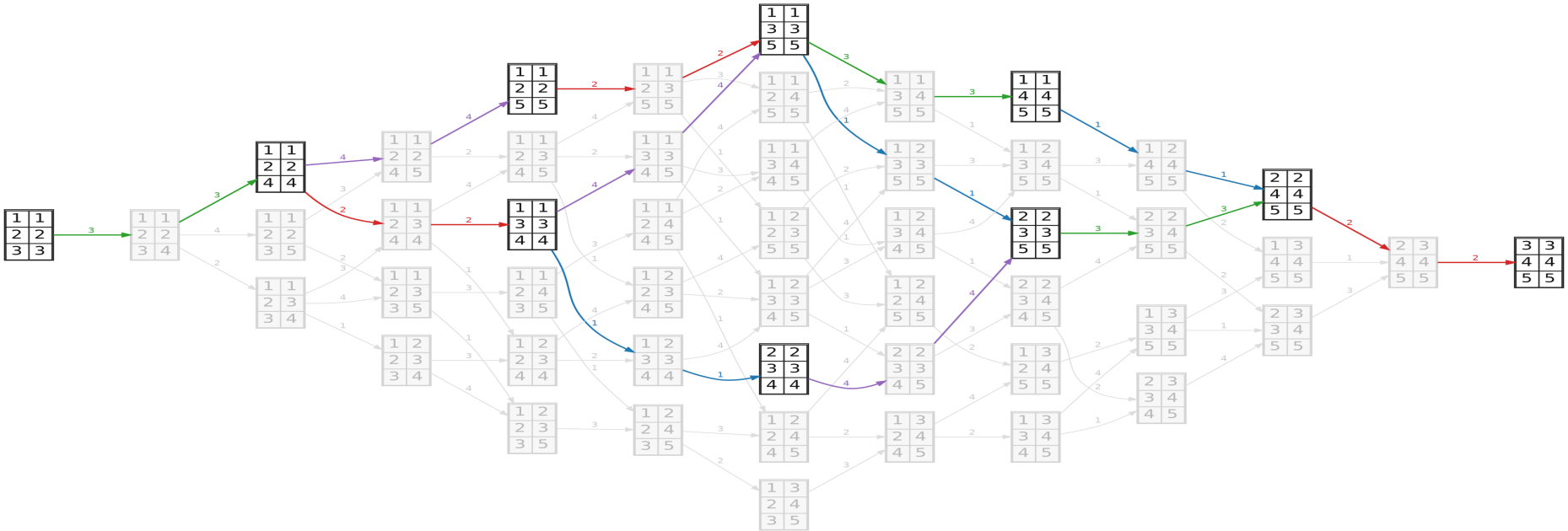
$m = 3 \downarrow$

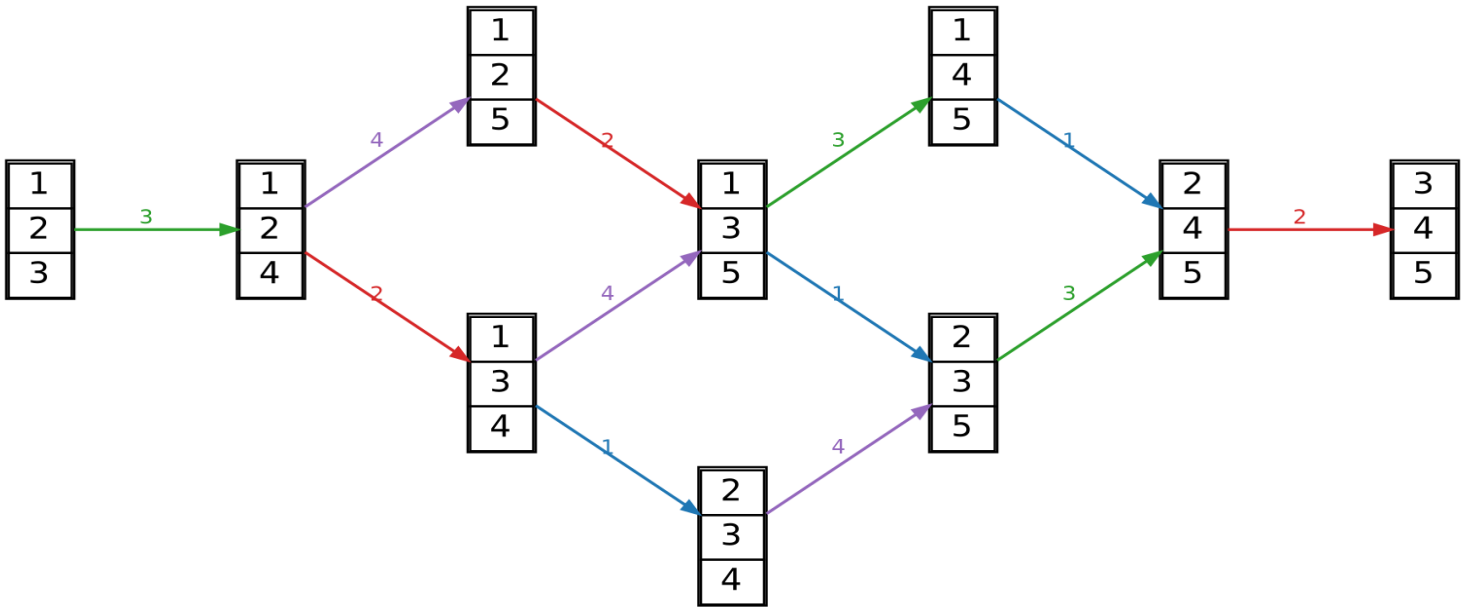
$V(6)$



Example (type A_4) $V(\varpi_3) \xrightarrow{m=2} V(2\varpi_3)$

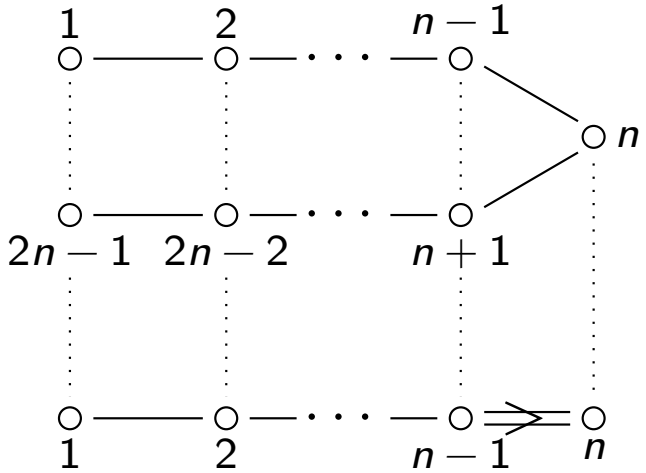






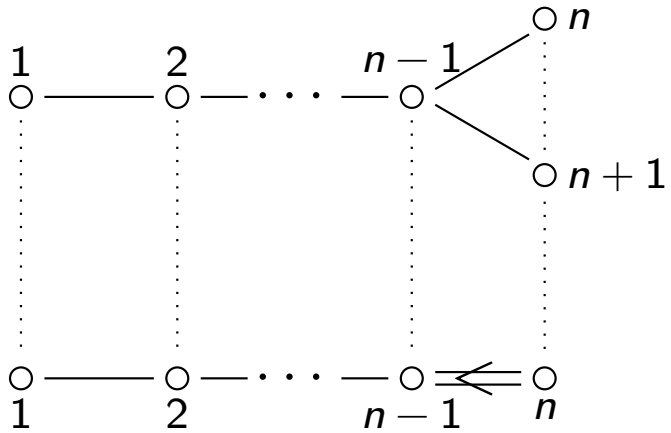
Folding of crystals (finite type)

A_{2n-1}



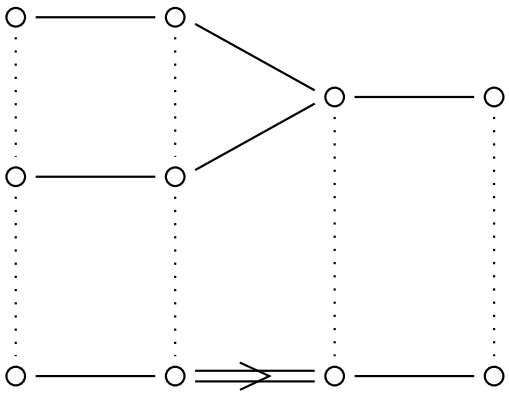
B_n

D_{n+1}



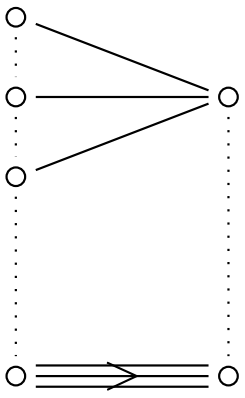
C_n

E_6

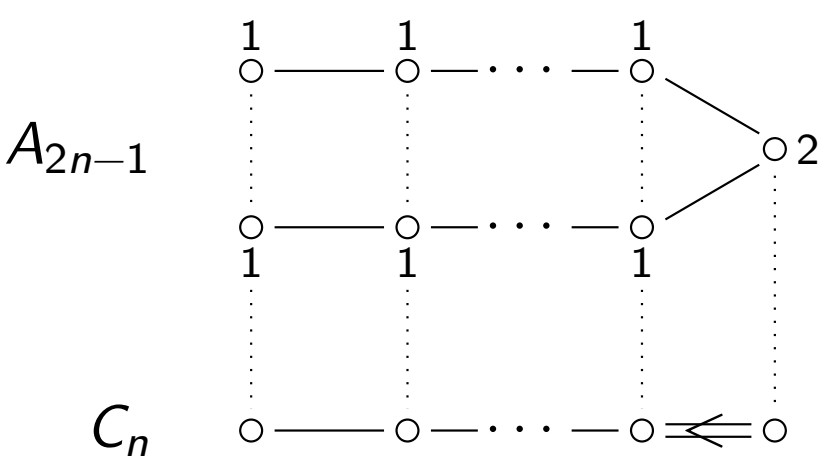


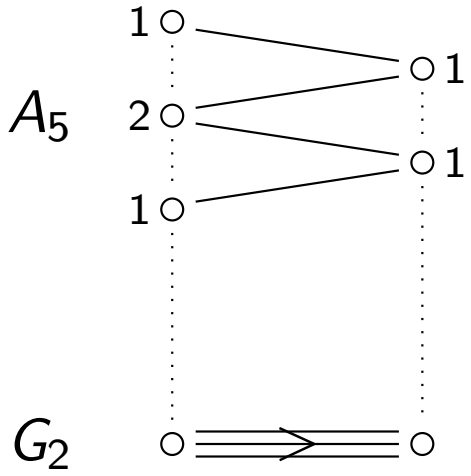
F_4

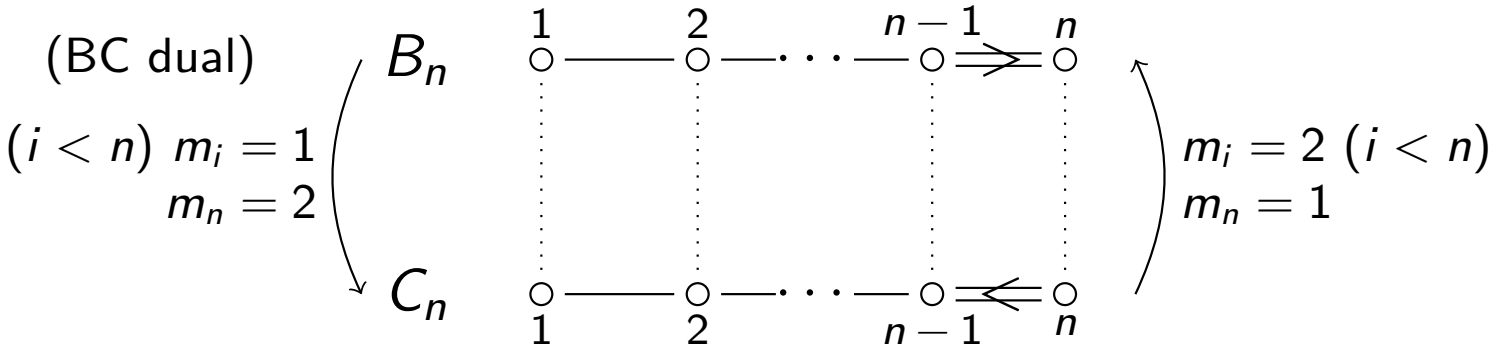
D_4



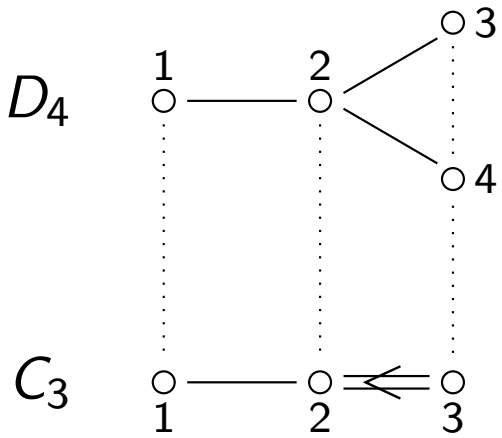
G_2


A_{2n-1}
C_n
1
1
1
2
1
1
1


A_5
G_2
1
2
1
1
1


(BC dual)
$(i < n)$ $m_i = 1$
$m_n = 2$
B_n
C_n
$m_i = 2$ $(i < n)$
$m_n = 1$
1
2
$n-1$
n
1
2
$n-1$
n

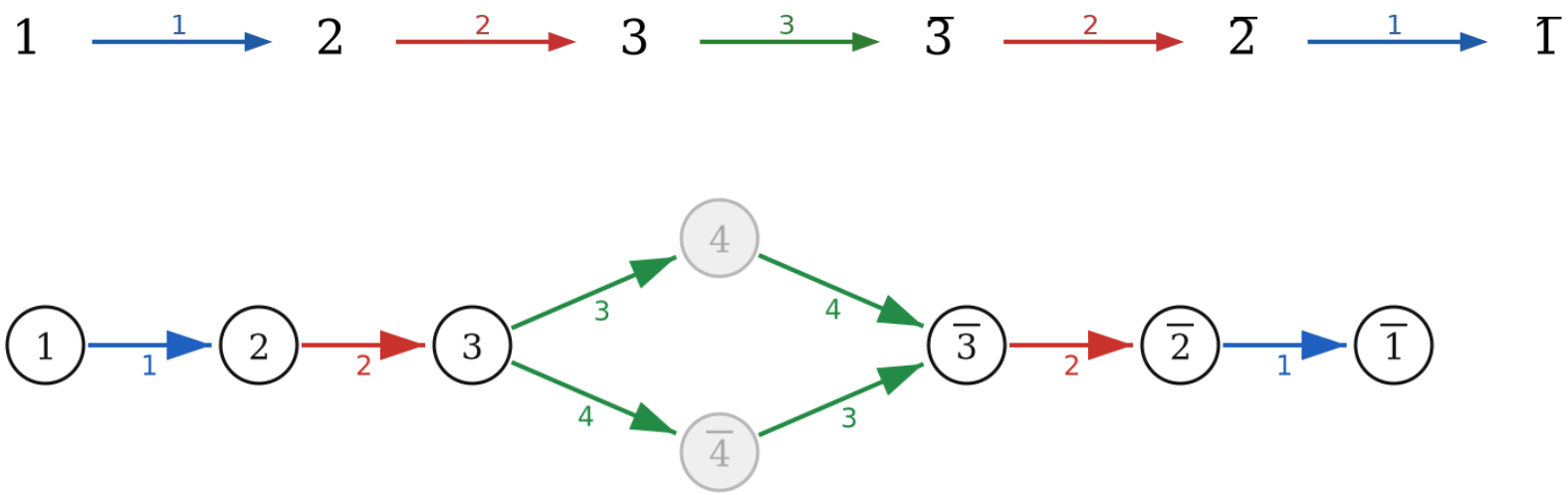
Example: $D_4 \Rightarrow C_3$



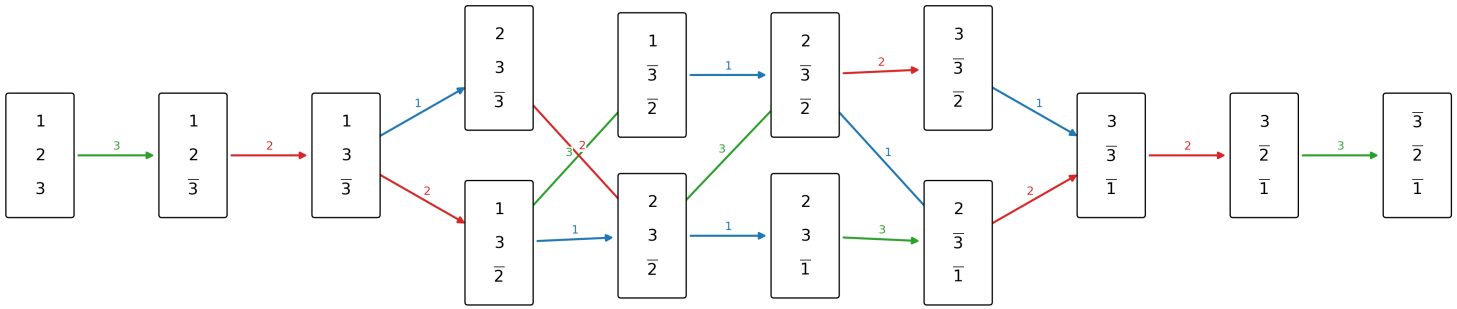
($*$ simply write $\widetilde{(\cdots)} = (\cdots)^C$)

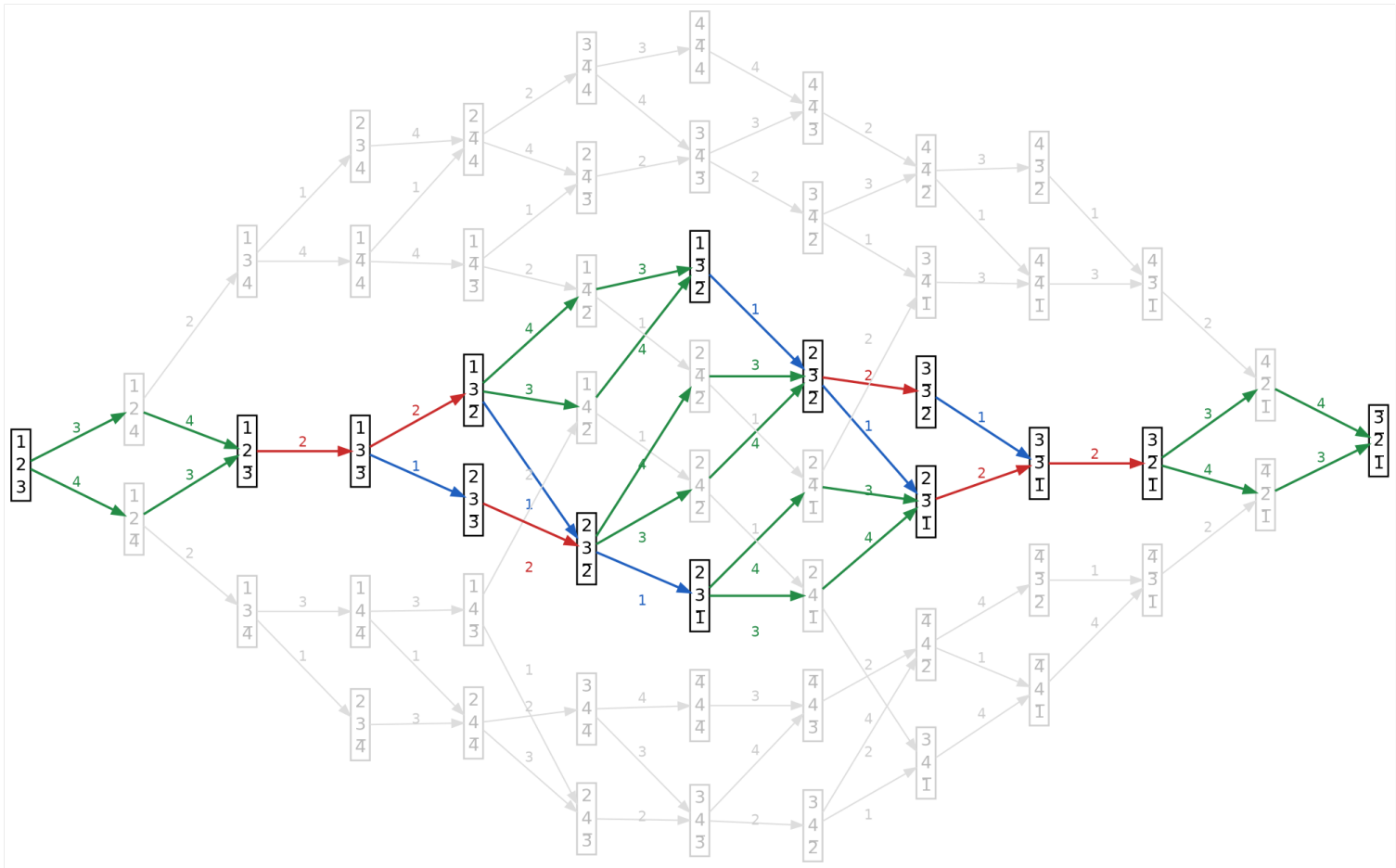
- $\alpha_1^C = \alpha_1,\ \alpha_2^C = \alpha_2,\ \alpha_3^C = \alpha_3 + \alpha_4$
- $\tilde{P} = \{\sum t_i \varpi_i \in P \mid t_3 = t_4\}$
- $h_1^C = h_1,\ h_2^C = h_2,\ h_3^C = h_3 (= h_4)$
- $\varpi_1^C = \varpi_1,\ \varpi_2^C = \varpi_2,\ \varpi_3^C = \varpi_3 + \varpi_4$
- $\tilde{X}_1^C = \tilde{x}_1,\ \tilde{X}_2^C = \tilde{x}_2,\ \tilde{X}_3^C = \tilde{x}_3 \tilde{x}_4$

Example $B(\varpi_1^C) \longrightarrow B(\varpi_1) : \mathrm{KN}_3^C(1) \longrightarrow \mathrm{KN}_3^D(1)$

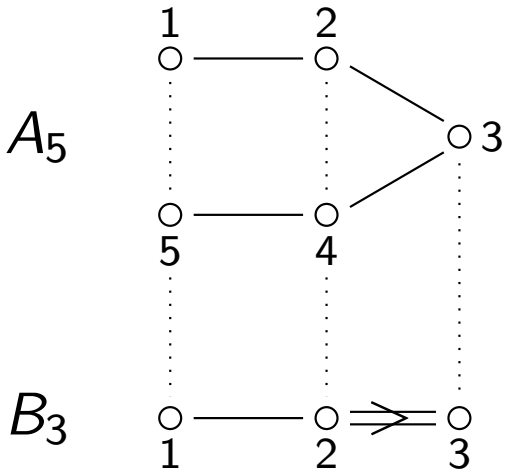


Example $B(\varpi_3^C) \longrightarrow B(\varpi_3 + \varpi_4) : \mathrm{KN}_3^C(1,1,1) \longrightarrow \mathrm{KN}_4^D(1,1,1)$





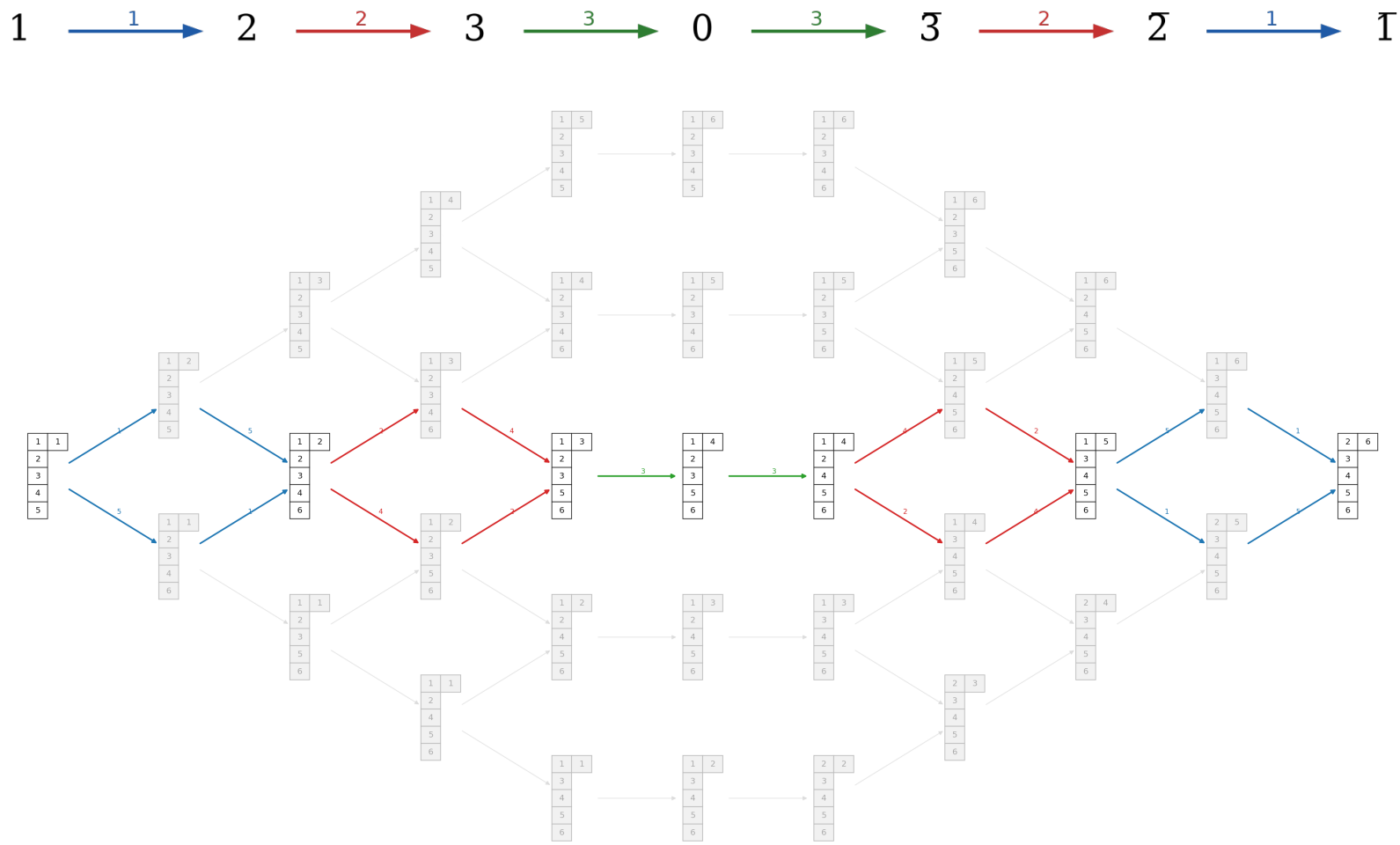
Example: $A_5 \Rightarrow B_3$



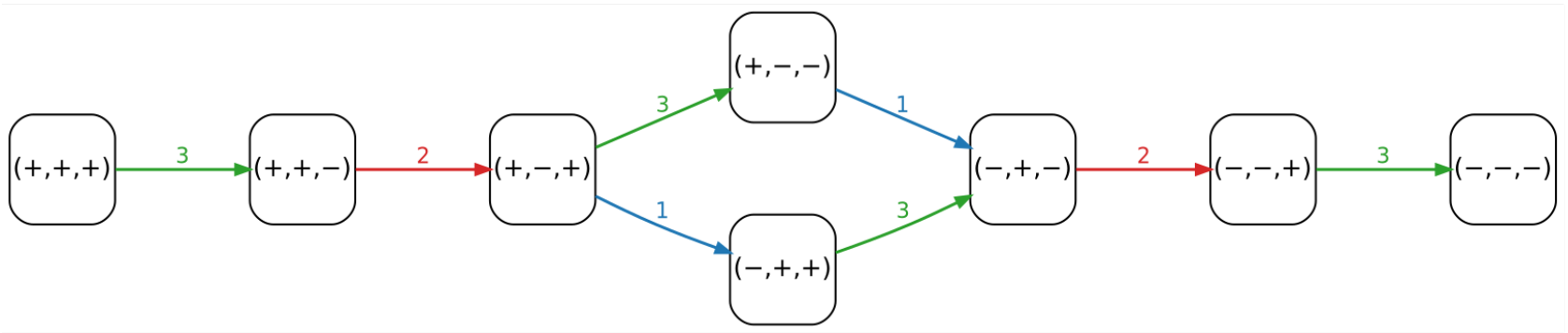
($*$ simply write $\widetilde{(\cdots)} = (\cdots)^B$)

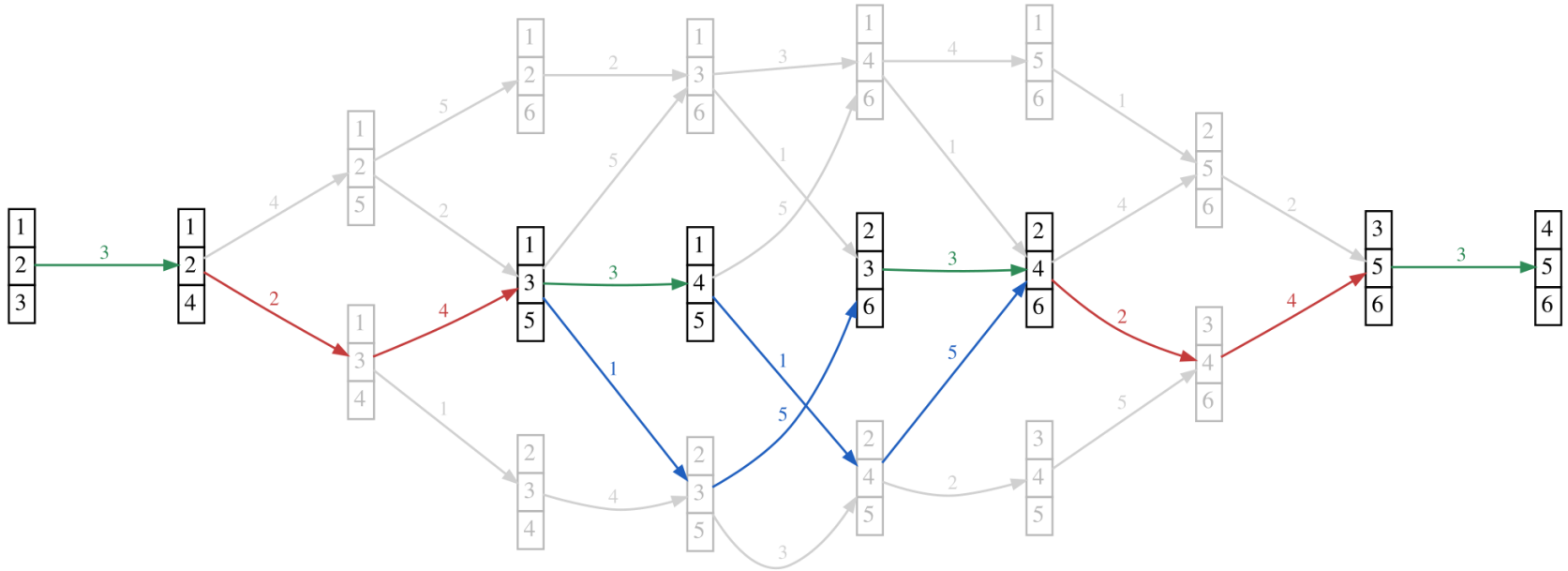
- $\alpha_1^B = \alpha_1 + \alpha_5,\ \alpha_2^B = \alpha_2 + \alpha_4,\ \alpha_3^B = \alpha_3$
- $\tilde{P} = \{\sum t_i \varpi_i \in P \mid t_1 = t_5, t_2 = t_4\}$
- $h_1^B = h_1 (= h_5),\ h_2^B = h_2 (= h_4),\ h_3^B = h_3$
- $\varpi_1^B = \varpi_1 + \varpi_5,\ \varpi_2^B = \varpi_2 + \varpi_4,\ \varpi_3^B = \varpi_3$
- $\tilde{X}_1^B = \tilde{x}_1 \tilde{x}_5,\ \tilde{X}_2^B = \tilde{x}_2 \tilde{x}_4,\ \tilde{X}_3^B = \tilde{x}_3$

Example $B(\varpi_1^B) \longrightarrow B(\varpi_5+\varpi_1) : \mathrm{KN}_3^B(1) \longrightarrow \mathrm{SST}_5(2,1,1,1,1)$

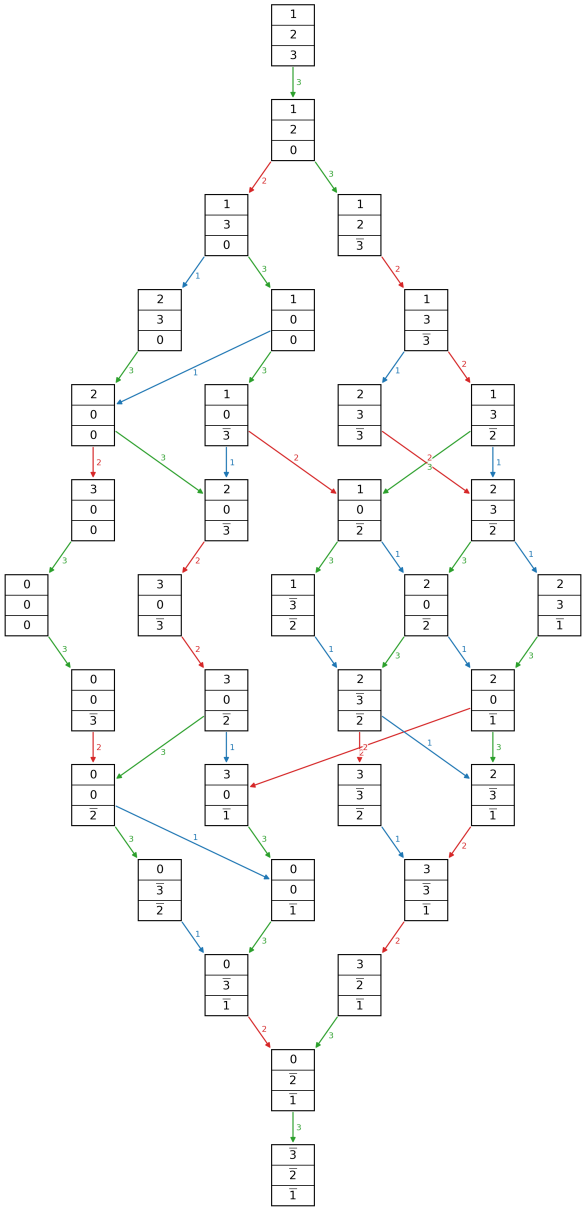


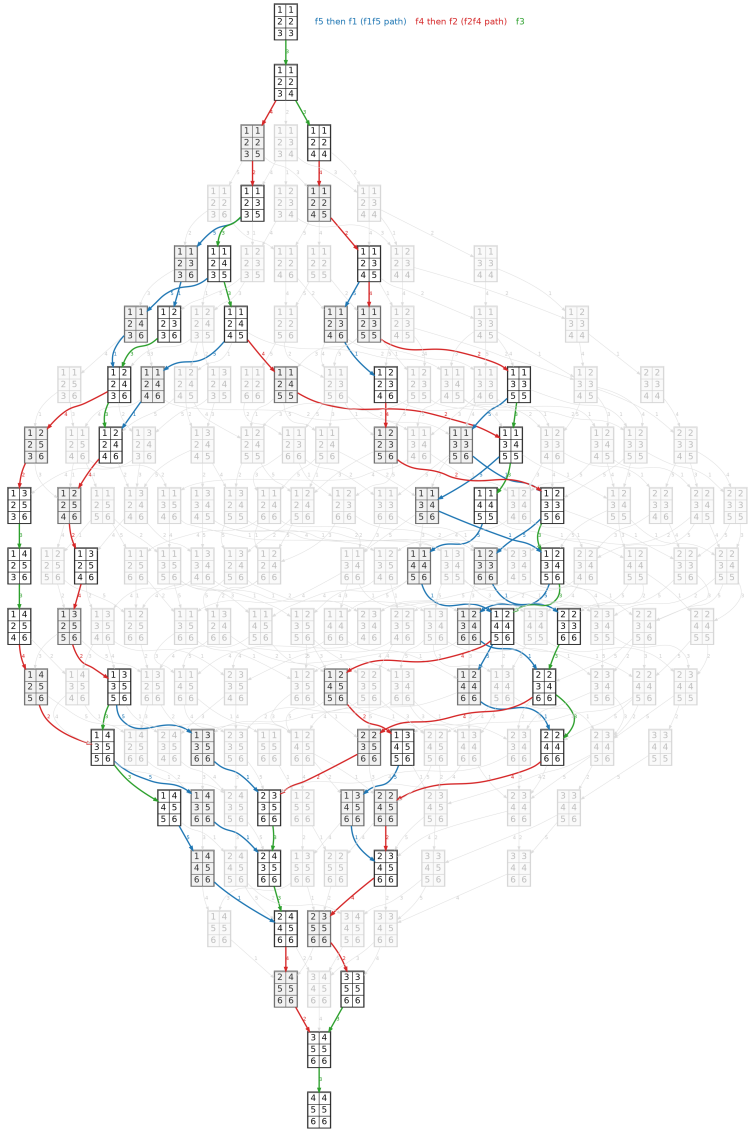
Example $B(\varpi_3^B) \longrightarrow B(\varpi_3) : B^{\mathrm{spin}} \longrightarrow \mathrm{SST}_5(1,1,1)$



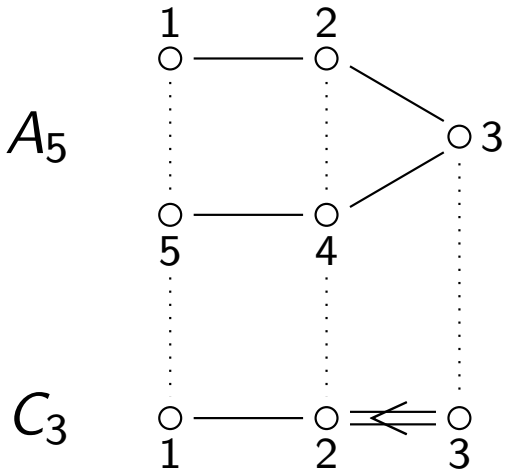


Example $B(2\varpi_3^B) \longrightarrow B(2\varpi_3) : \mathrm{KN}_3^B(1,1,1) \longrightarrow \mathrm{SST}_5(2,2,2)$





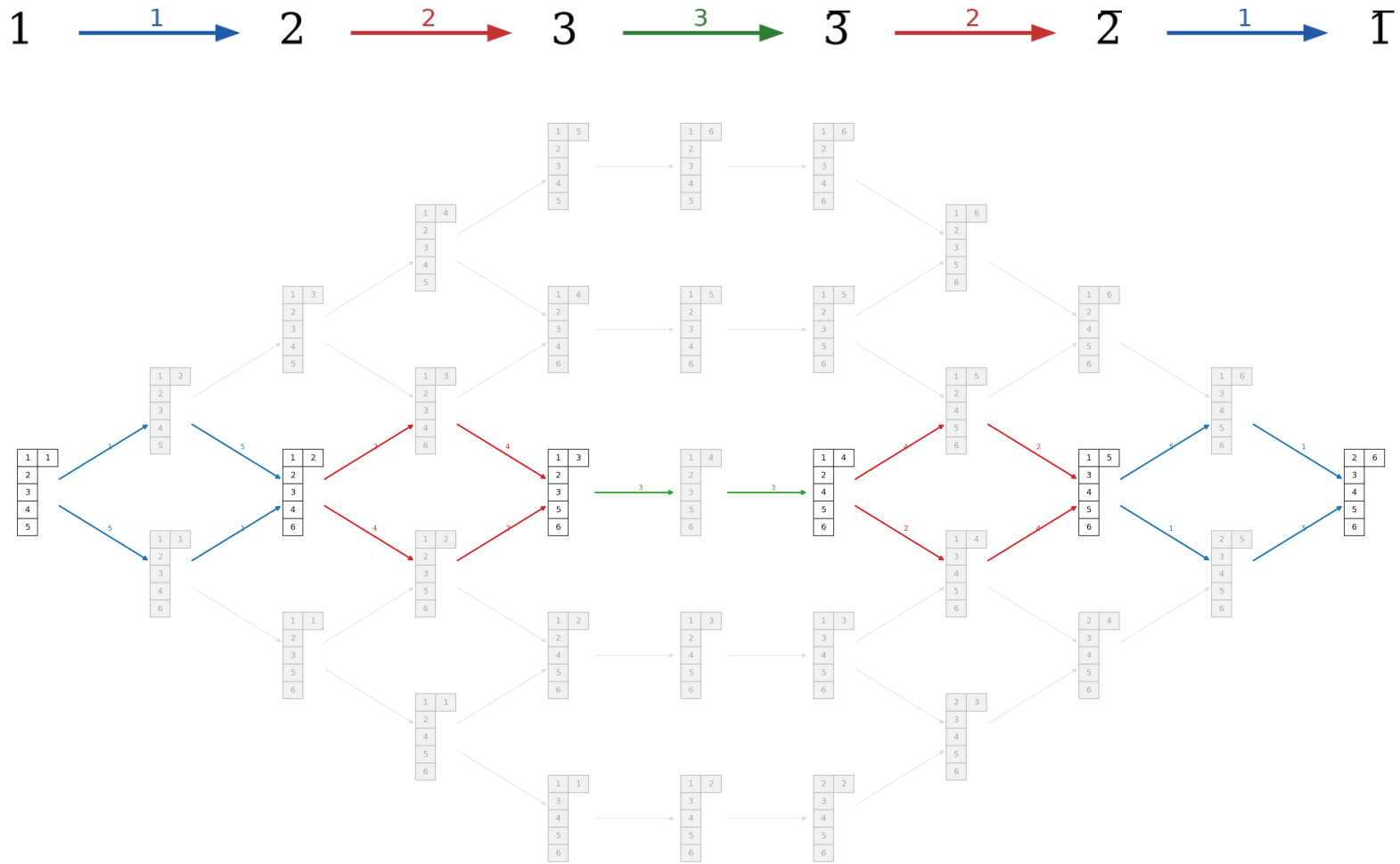
Example: $A_5 \Rightarrow C_3$



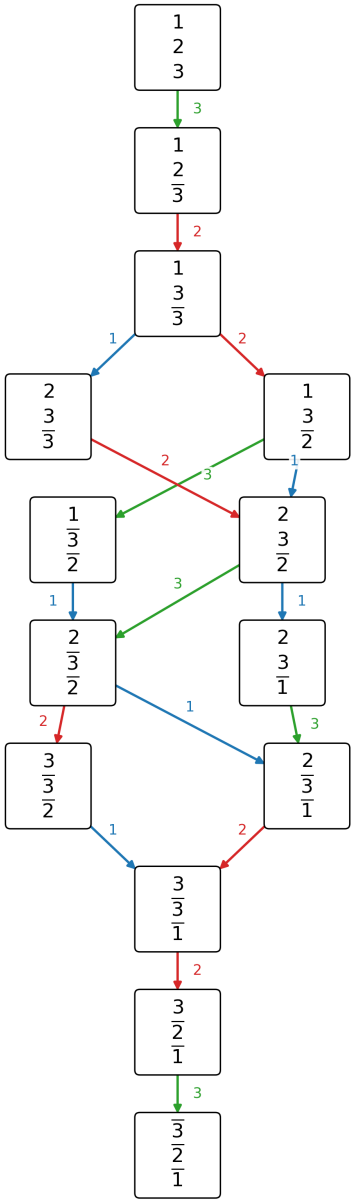
($*$ simply write $\widetilde{(\cdots)} = (\cdots)^C$)

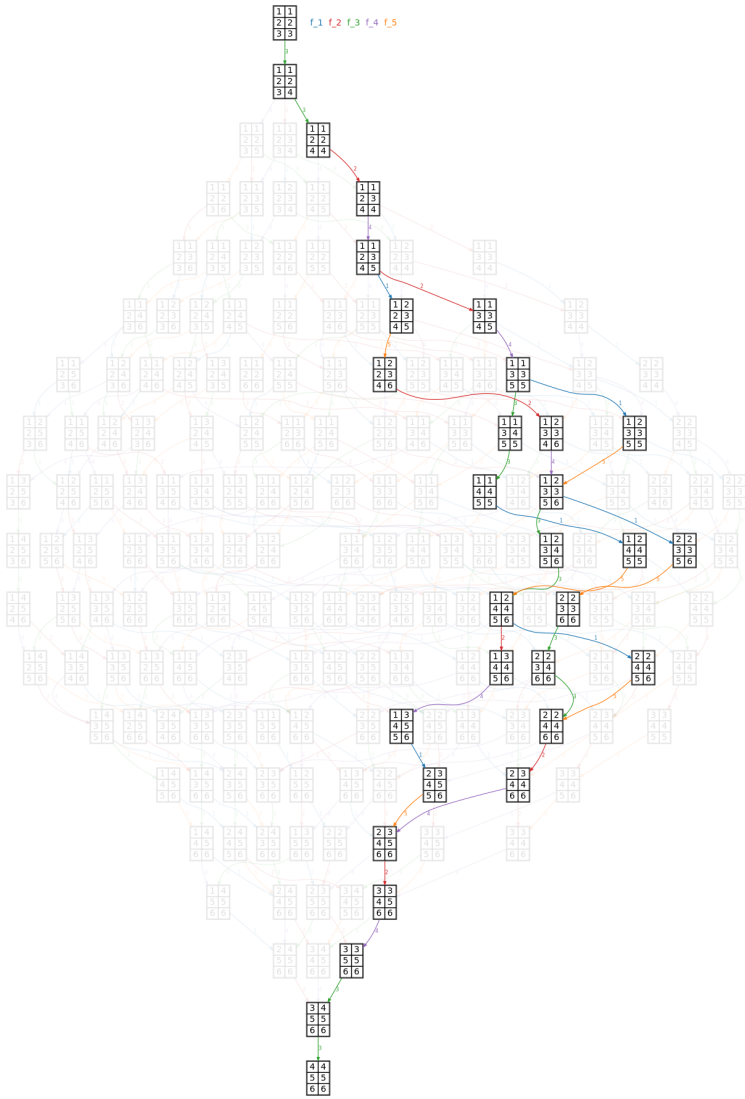
- $\alpha_1^C = \alpha_1 + \alpha_5,\ \alpha_2^C = \alpha_2 + \alpha_4,\ \alpha_3^C = 2\alpha_3$
- $\tilde{P} = \{\sum t_i \varpi_i \in P \mid t_1 = t_5, t_2 = t_4, t_3 \in 2\mathbb{Z}\}$
- $h_1^C = h_1 (= h_5),\ h_2^C = h_2 (= h_4),\ h_3^C = \frac{1}{2} h_3$
- $\varpi_1^C = \varpi_1 + \varpi_5,\ \varpi_2^C = \varpi_2 + \varpi_4,\ \varpi_3^C = 2\varpi_3$
- $\tilde{X}_1^C = \tilde{x}_1 \tilde{x}_5,\ \tilde{X}_2^C = \tilde{x}_2 \tilde{x}_4,\ \tilde{X}_3^C = \tilde{x}_3^2$

Example $B(\varpi_1^C) \longrightarrow B(\varpi_5 + \varpi_1) : \mathrm{KN}_3^C(1) \longrightarrow \mathrm{SST}_5(2,1,1,1,1)$



Example $B(\varpi_3^C) \longrightarrow B(2\varpi_3) : \mathrm{KN}_3^C(1,1,1) \longrightarrow \mathrm{SST}_5(2,2,2)$





Contents

1. Background: Kashiwara embedding
2. Main result: similarity and folding crystals
3. Fixed point subset
4. Virtual crystals
5. Application: non-simply-laced KR crystals

Fixed point subset

Without stated, we assume $m_i = 1$ for all $i \in I$ from now on.

Q) Describe the image of an injective map S_λ.

A) $(\mathbb{Z}^\infty_{\underline{i}})^0$: fixed point subset [Naito-Sagaki, 2004]

Setting

· $w : I \to I$: Dynkin diagram automorphism, i.e., $a_{ij} = a_{w(i),w(j)}$

· J : the set of w-orbits in I
(or consider $J \subseteq I$ by taking representatives of orbits)

· $\xi : I \twoheadrightarrow J$ with $\xi(i) = (\text{orbit of } i \in I)$

· $N_i = |\xi(i)|$: order of $i \in I$

· $\tilde{I}_j := (w^{N_j-1}(j), \ldots, w^2(j), w(j), j)$ for $j \in J$

· $w(\underline{i}) := (\ldots, w(i_2), w(i_1))$ for $\underline{i} = (\ldots, i_2, i_1) \in I^\infty$

Prop [Nakashima, 1999] For $i, j \in I$ with $a_{ij} = 0$,

$\exists$ crystal isomorphism $\phi^{(0)} : \mathbb{B}_i \otimes \mathbb{B}_j \xrightarrow{\sim} \mathbb{B}_j \otimes \mathbb{B}_i, \quad (x, y) \mapsto (y, x).$

· ϕ_j : the crystal isomorphism from $\mathbb{B}_{w(\tilde{i}_j)}$ to $\mathbb{B}_{\tilde{i}_j}$ via $\phi^{(0)}$

e.g. $(N_j = 3)$

$$\phi_j : \mathbb{B}_j \otimes \mathbb{B}_{w^2(j)} \otimes \mathbb{B}_{w(j)} \longrightarrow \mathbb{B}_{w^2(j)} \otimes \mathbb{B}_{w(j)} \otimes \mathbb{B}_j$$

$$\phi_j : (x, y, z) \underset{\phi_{23}^{(0)}}{\mapsto} (y, x, z) \underset{\phi_{12}^{(0)}}{\mapsto} (y, z, x)$$

Now, suppose $\underline{i}$ is of the form $(\ldots, \tilde{I}_{j_2}, \tilde{I}_{j_1})$ for $j_k \in J$ with $j_k \neq j_{k+1}$.

· $\widehat{\underline{i}} := (\ldots, j_2, j_1) \in J^\infty$

· $\phi := \cdots \otimes \phi_{j_2} \otimes \phi_{j_1} : \mathbb{B}_{w(\underline{i})} \to \mathbb{B}_{\underline{i}}$

· w-action on $\mathbb{Z}^\infty_{\underline{i}}$, denoted by w (notation abuse)

$w := \phi \circ \mathrm{id}_w$ where $\mathrm{id}_w : \mathbb{Z}^\infty_{\underline{i}} \to \mathbb{Z}^\infty_{w(\underline{i})}$ is the identity map on $\mathbb{Z}^\infty$

Let $(\mathbb{Z}^{\infty}_{\underline{i}})^0$ be the fixed point subset of $\mathbb{Z}^{\infty}_{\underline{i}}$ under the w-action.

Proposition (Naito-Sagaki, 2004)

The fixed point subset $(\mathbb{Z}^{\infty}_{\underline{i}})^0$ *is described as follows:*

$$(\mathbb{Z}^{\infty}_{\underline{i}})^0 = \{(\ldots, \tilde{l}_{j_2}(x_2), \tilde{l}_{j_1}(x_1)) \in \mathbb{Z}^{\infty}_{\underline{i}} \mid j_k \in J, x_k \in \mathbb{Z}, x_k = 0 \text{ for } k \gg 0\},$$

where $\tilde{l}_j(n) = (\underbrace{n, n, \ldots, n}_{N_j \text{ times}})$ *for* $n \in \mathbb{Z}, j \in J$.

Define w-crystal operators on $(\mathbb{Z}^{\infty}_{\underline{i}})^0$ by

$$\tilde{X}_j = \prod_{k=0}^{N_j-1} \tilde{x}_{w^k(j)} \quad \text{for } x \in \{e, f\},\ j \in J.$$

Theorem (Naito-Sagaki, 2004)

The set $(\mathbb{Z}^{\infty}_{\underline{i}})^0 \cup \{0\}$ is stable under $\tilde{X}_j$ for $j \in J$. Moreover, the map $\Phi : (\mathbb{Z}^{\infty}_{\underline{i}})^0 \to \widehat{\mathbb{Z}}^{\infty}_{\widehat{\underline{i}}}$ given by

$$\Phi : (\ldots, \tilde{I}_{j_2}(x_2), \tilde{I}_{j_1}(x_1)) \mapsto (\ldots, x_2, x_1)$$

is a (weight-preserving) bijection such that $\Phi \circ \widetilde{X}_j = \hat{x}_j \circ \Phi$.

As a corollary, we obtain

$$\Psi^I : \mathrm{im}(S_\lambda : B^J(\lambda) \to B^I(\lambda)) \xrightarrow{\sim} (\mathbb{Z}^{\infty}_{\underline{i}})^0 \cap \Psi^I(B^I(\lambda)).$$

Proof

1. Φ is a bijection : clear
2. f-strict $\Rightarrow$ e-strict : standard argument
3. strict $\Rightarrow$ stable : $\tilde{X}_j b = \Phi^{-1}(\hat{f}_j \Phi(b)) \in (\mathbb{Z}^{\infty}_{\underline{i}})^0$ $(\because \hat{x}_j \Phi(b) \in \widehat{\mathbb{Z}}^{\infty}_{\underline{i}})$

   ETS Φ : f-strict
4. f-strict : keep track of $\sigma^{(i)}$ and $\min M^{(i)}, \max M^{(i)}$.

Remark What about m-fold scaling?

$$\boxed{\Psi : \mathrm{im}(S_m) \xrightarrow{\sim} m\mathbb{Z}^{\infty} \cap \Psi(B(\lambda))}$$

$(\because \varepsilon_i(S_m(b)) = m\varepsilon_i(b) +$ defining inequalities are linear)

Contents

1. Background: Kashiwara embedding
2. Main result: similarity and folding crystals
3. Fixed point subset
4. Virtual crystals
5. Application: non-simply-laced KR crystals

General setting

Recall that $\xi : I \twoheadrightarrow J$ induces a map

$$\tilde{\xi} : \tilde{P} \to P, \quad \tilde{\omega}_i \mapsto \sum_{j \in \xi^{-1}(i)} m_i \omega_i.$$

For $j \in J$, define

$$\hat{x}_j := \prod_{i \in \xi^{-1}(j)} \tilde{x}_i^{m_i} \ (= \tilde{X}_j) \text{ for } x \in \{e, f\} : \text{virtual crystal operators} \tag{1}$$

$$\hat{\varepsilon}_j = \frac{1}{m_i} \varepsilon_i, \ \hat{\varphi}_j = \frac{1}{m_i} \varphi_i \text{ for any } i \in \xi^{-1}(j) \tag{2}$$

$$\widehat{\mathrm{wt}} = \tilde{\xi}^{-1} \circ \mathrm{wt} \tag{3}$$

Virtual crystals

Definition (Pan-Scrimshaw 2018 (cf. Okado-Schilling-Shimozono 2003))

Let B be a I-crystal and $V \subseteq B$ be a set.

A virtual crystal is the quadruple $(V, B, \xi, (m_i))$ such that V has a J-crystal structure defined by (1), (2), and (3).

We say $\hat{B}$ virtualizes in B if there exists a J-crystal isomorphism

$$v : \hat{B} \to V,$$

and the resulting isomorphism v is called the virtualization map.

Results for the virtualization

- The similarity embedding S_λ is the virtualization map.
- Some type-dependent results are also known; splitting KN tableaux
- construction of KR crystals : key results (see next section)

- LS path : [Pan-Scrimshaw 2018]
- Nakajima monomial : [Salisbury-Scrimshaw 2018]
  Result $\mathrm{im}(S_\lambda) = \{$ orbit invariant, m-divisibility on exponents $\}$

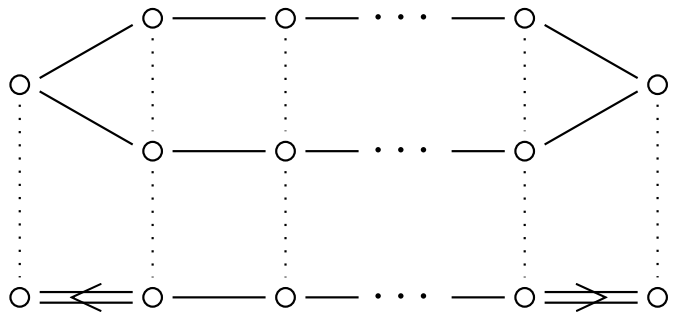
Contents

1. Background: Kashiwara embedding
2. Main result: similarity and folding crystals
3. Fixed point subset
4. Virtual crystals
5. Application: non-simply-laced KR crystals

Folding of crystals (affine type)

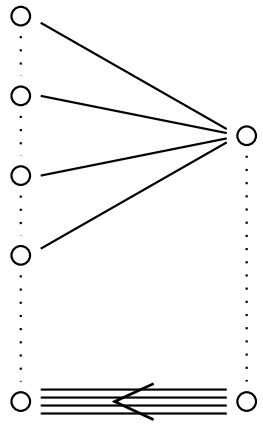
$A^{(1)}_{2n-1}$

$D^{(2)}_{n}$



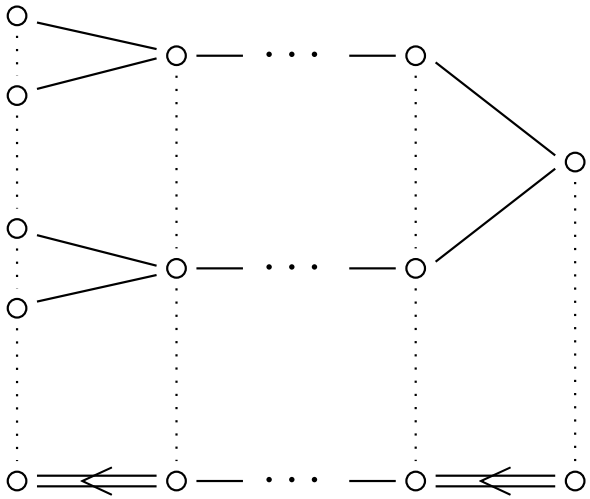
$D^{(1)}_{4}$

$A^{(2)}_{2}$



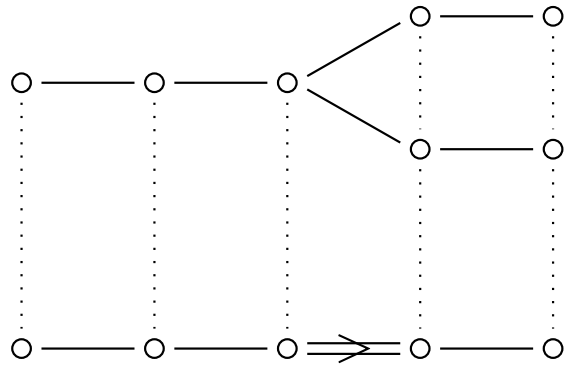
$D^{(1)}_{2n+2}$

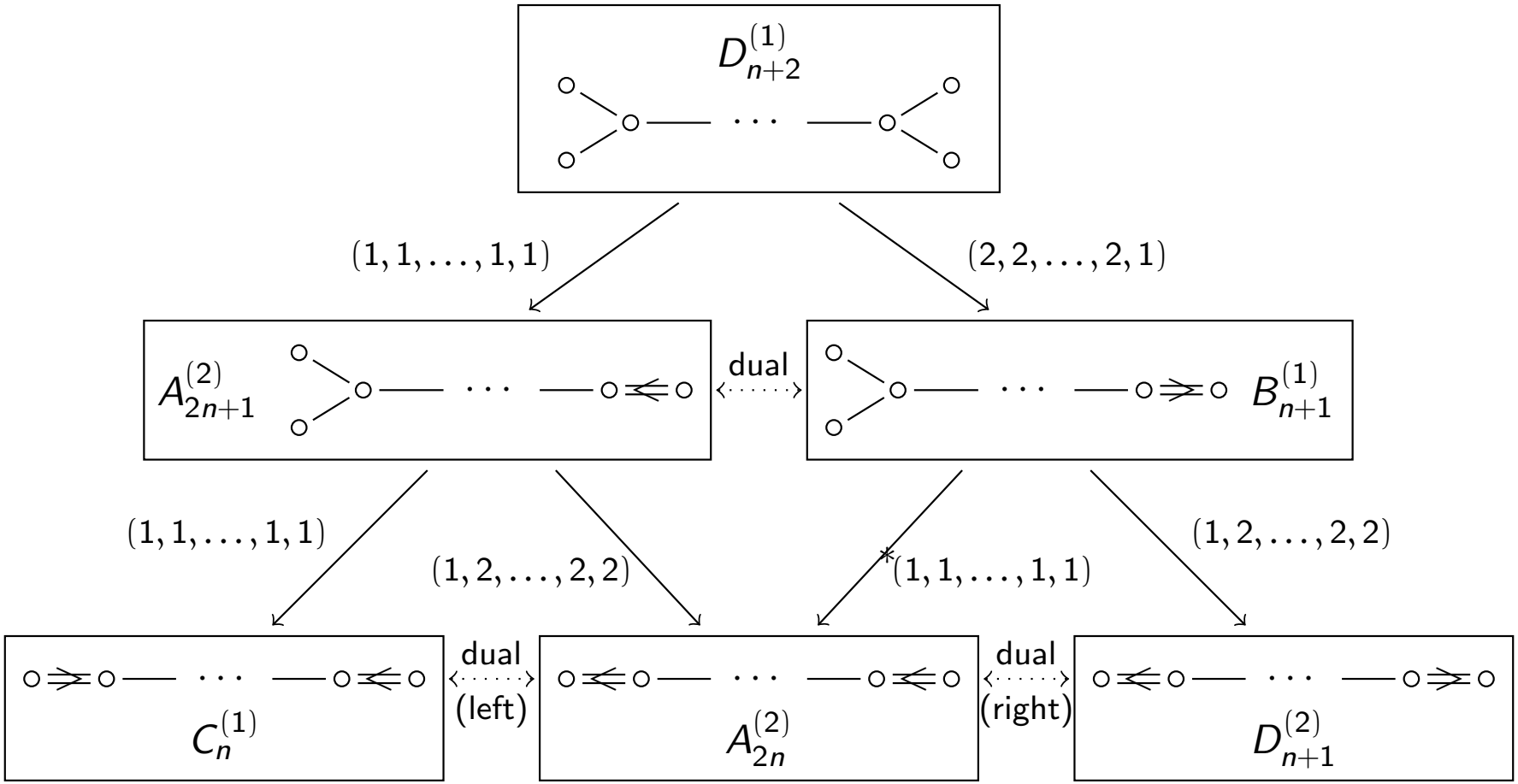
$A^{(2)}_{2n}$



$E^{(1)}_{6}$

$E^{(2)}_{6}$




$D_{n+2}^{(1)}$
$(1,1,\ldots,1,1)$
$(2,2,\ldots,2,1)$
$A_{2n+1}^{(2)}$
dual
$B_{n+1}^{(1)}$
$(1,1,\ldots,1,1)$
$(1,2,\ldots,2,2)$
$(1,1,\ldots,1,1)$
$(1,2,\ldots,2,2)$
$C_n^{(1)}$
dual
(left)
$A_{2n}^{(2)}$
dual
(right)
$D_{n+1}^{(2)}$

Kirillov-Reshetikhin (KR) modules

$*$ $\mathfrak{g}$: affine Kac-Moody algebra with index I

$\cdot$ KR modules $W_s^{(r)}$ $(r \in I_0, s \in \mathbb{Z}_{>0})$
: finite dimensional irreducible $U_q'(\mathfrak{g})$-module with the Drinfeld polynomial

$$P_j(u) = \begin{cases} (1 - a_r q_r^{1-s} u)(1 - a_r q_r^{3-s} u) \dots (1 - a_r q_r^{s-1} u) & \text{if } j = r \\ 1 & \text{if } j \neq r \end{cases}$$

Theorem (Okado-Schilling 2008)

The Kirillov-Reshetikhin module $W_s^{(r)}$ associated to any non-exceptional affine Kac-Moody algebra has a crystal basis $B^{r,s}$.

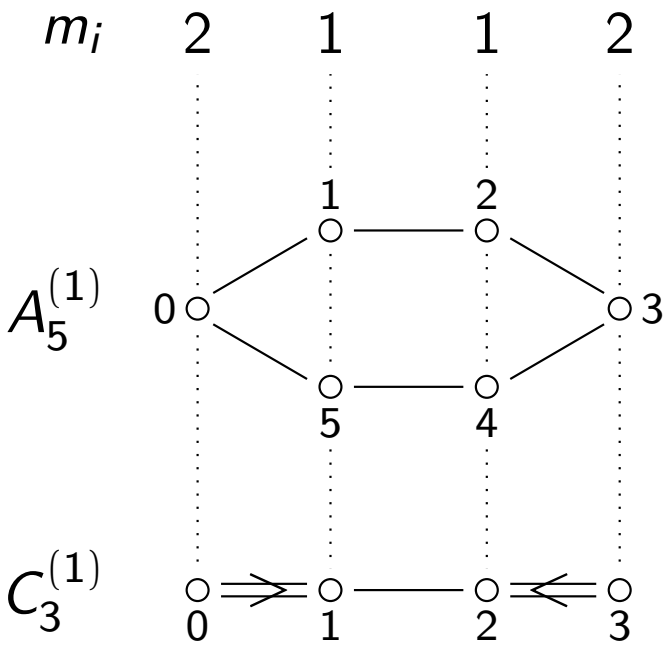
Construction of KR crystals

Main Ref
1. Okado-Schilling-Shimozono, 2003
2. Fourier-Okado-Schilling, 2009

Strategy for constructing KR crystals

- Determine the decomposition $B^{r,s} \cong \bigoplus B_{\mathfrak{g}_{\mathrm{fin}}}(\lambda)$ as $\mathfrak{g}_{\mathrm{fin}}$-crystal
- Construct the 0-action (via promotion or virtualization)
- Prove that the resulting crystal is isomorphic to the KR crystal (conjectural in some cases of OSS, established in FOS)

KR crystal example: $B^{1,1}$ of type $C_3^{(1)}$



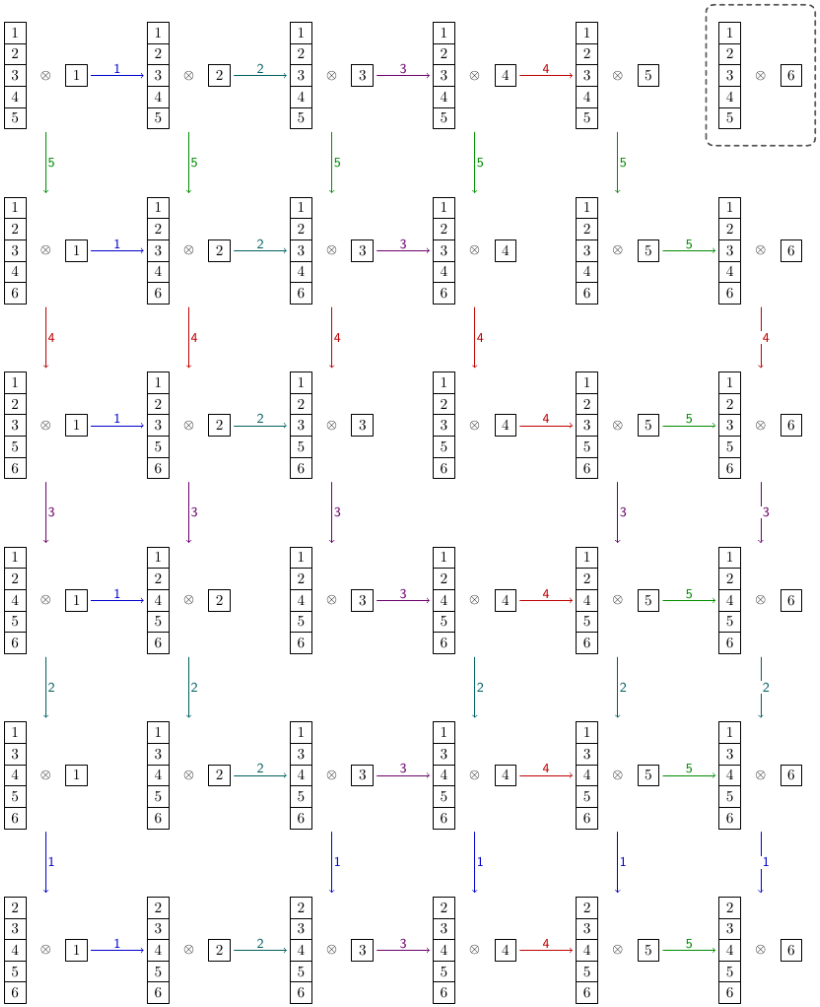
$$B_C(\varpi_1) \xrightarrow{\text{virtualization}} B_A(\varpi_5) \otimes B_A(\varpi_1)$$

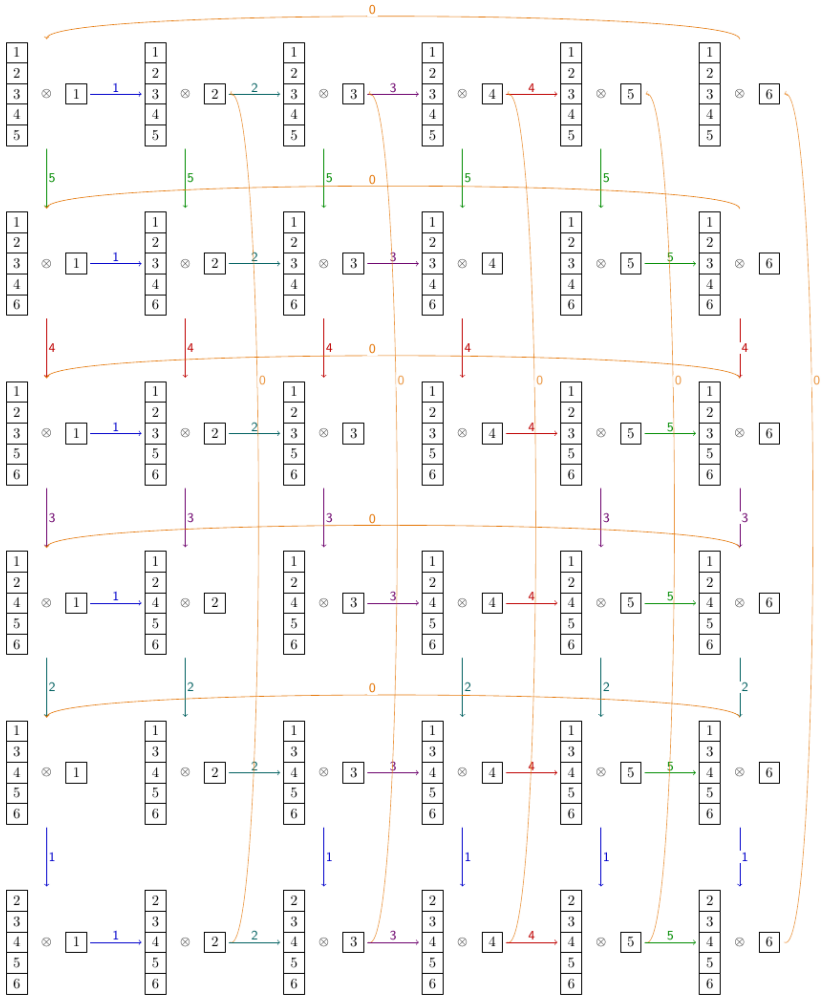
$$\mathfrak{g}_{\text{fin}} \uparrow \qquad \uparrow \mathfrak{g}_{\text{fin}}$$

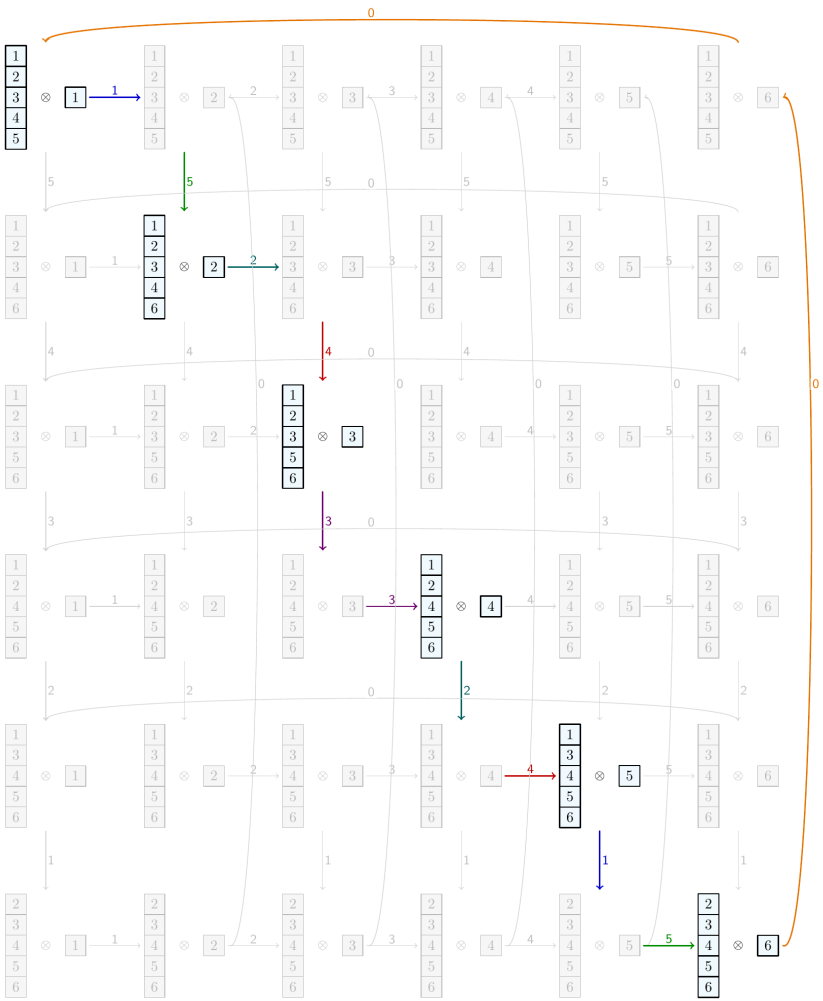
$$B_C^{1,1} \xrightarrow[\text{virtualization}]{} B_A^{5,1} \otimes B_A^{1,1}$$

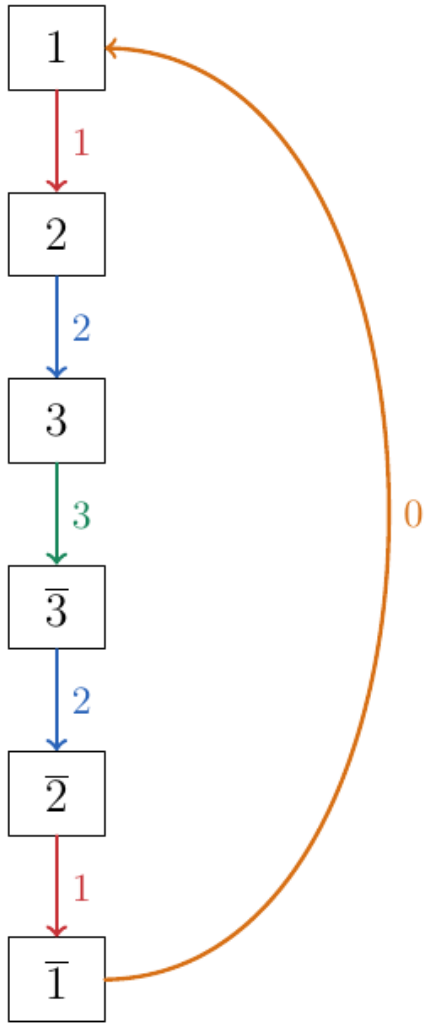
$$\tilde{X}_0^C = \tilde{x}_0^2, \qquad \tilde{X}_1^C = \tilde{x}_1\tilde{x}_5$$

$$\tilde{X}_2^C = \tilde{x}_2\tilde{x}_4, \quad \tilde{X}_3^C = \tilde{x}_3^2$$








1
1
2
2
3
3
$\overline{3}$
2
$\overline{2}$
1
$\overline{1}$
0

References I


Ghislain Fourier, Masato Okado, and Anne Schilling.
Kirillov-Reshetikhin crystals for nonexceptional types.
Adv. Math., 222(3):1080–1116, 2009.


Masaki Kashiwara.
The crystal base and Littelmann's refined Demazure character formula.
Duke Math. J., 71(3):839–858, 1993.


Masaki Kashiwara.
Similarity of crystal bases.
In *Lie algebras and their representations (Seoul, 1995)*, volume 194 of *Contemp. Math.*, pages 177–186. Amer. Math. Soc., Providence, RI, 1996.


Satoshi Naito and Daisuke Sagaki.
A rationalization of the crystal $\mathbb{Z}^\infty$ and a diagram automorphism.
J. Pure Appl. Algebra, 189(1-3):279–295, 2004.

References II


Toshiki Nakashima.
Polyhedral realizations of crystal bases and braid-type isomorphisms.
In *Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998)*, volume 248 of *Contemp. Math.*, pages 419–435. Amer. Math. Soc., Providence, RI, 1999.


Toshiki Nakashima and Andrei Zelevinsky.
Polyhedral realizations of crystal bases for quantized Kac-Moody algebras.
Adv. Math., 131(1):253–278, 1997.


Masato Okado and Anne Schilling.
Existence of Kirillov-Reshetikhin crystals for nonexceptional types.
Represent. Theory, 12:186–207, 2008.

References III


Masato Okado, Anne Schilling, and Mark Shimozono.
Virtual crystals and fermionic formulas of type $D_{n+1}^{(2)}, A_{2n}^{(2)}$, and $C_n^{(1)}$.
Represent. Theory, 7:101–163, 2003.


Jianping Pan and Travis Scrimshaw.
Virtualization map for the Littelmann path model.
Transform. Groups, 23(4):1045–1061, 2018.


Ben Salisbury and Travis Scrimshaw.
Virtual crystals and Nakajima monomials.
SIGMA Symmetry Integrability Geom. Methods Appl., 14:Paper No. 103, 19, 2018.