Orbifolds in Conformal field theories

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Abstract

We first review the basics of conformal field theories, and get a few results on 2D conformal field theories. After defining the simplest exemple of a conformal field theory, the free boson, we define correlation functions, see how conformal invariance puts constraints on these functions, and compute the correlation functions for the free boson. We then see the radial quantization, which is the first step towards orbifolds. Doing so, we study the free boson on the cylinder. Next, we mathematically construct orbifolds, and see how they are used to construct new conformal field theories from old ones with global symmetries. We see the importance of the modular transformations in the process, and study conformal field theories on the torus. We finally study the free boson on the torus, and compute two different orbifolds for the free boson.

Contents

Chapter 1

Conformal field theories

This section serves as a short introduction to conformal field theories. It is mostly based on [FMS97]. In 1.1, we will first recall what is a quantum field theory, and define what is a conformal field theory from it. In 1.2, we will see how the group of conformal transformations impose strong constraints on the theory. Finally, in 1.3 we will see the simplest exemple, that of the free boson.

1.1 Introduction

1.1.1 Beyond quantum mechanics

Quantum field theory (QFT) was invented to go beyond quantum mechanics. Strictly speaking, quantum mechanics and quantum field theory are equivalent: one can recover the same equations of motion from both theories. However, quantum field theory tries to overcome many complications of quantum mechanics, to make computations as intuitive as possible. It is the product of many refinements of quantum mechanics. Therefore, let's see the limitations that led to these refinements.

A first limitation of quantum mechanics, as they were first formulated, is that it can be bothersome to create or annihilate particles, because one state describes one particle. To overcome this issue, physicists developped the second quantization. In this formalism, instead of describing only one particle, states describe the whole system. If $\mathcal{H}$ is the Hilbert space spanning all possible states for a particle, the newly considered Hilbert space should look something like $\oplus_{n\in\mathbb{N}}\mathcal{H}_*^{\oplus n}$, where $\mathcal{H}_*^{\oplus n}$ is the Hilbert space describing a system of n particles. We will not go in details here on the construction of this space, called the Fock space. A detailed construction of the Bosonic Fock space can be found here [Cha18]. Replacing $\mathcal{H}$ by the Fock space, we are able to use 2 new operators, the creation operator $a^\dagger$ and the annihilation operator a. As indicated by their names, the creation operator creates a particle, whilst the annihilation operator annihilates a particle. Of course, there should be as many such operators as number of possible states for the particle.

Another limitation of quantum mechanics is that its axioms do not inherently respect special relativity, meaning the theory is not explicitely Poincaré covariant. In its classical formulation, the states clearly depend of time. To overcome this, we need to go from the Schrödinger picture to the Heisenberg picture. Classically,

$$\psi_t = e^{-itH}\psi \tag{1.1}$$

This perspective is called the Schrödinger picture. But instead of having time-independent operators along with states describing the particle at a given time, we can suppose that a state describes the particle across the whole spacetime, whilst operators are time-dependant and return, for exemple the momentum, at a given time. We then have

$$A_t = e^{itH}Ae^{-itH} \tag{1.2}$$

This perspective is called the Heisenberg picture.

These two pictures are strictly equivalents. When evaluating a state against an operator, we have

$$(\psi_t, A\psi_t) = (e^{-itH}\psi, Ae^{-itH}\psi) = (\psi, e^{itH}Ae^{-itH}\psi) = (\psi, A_t\psi) \tag{1.3}$$

Putting the second quantization together with the Heisenberg picture, we can have states describing the system across the whole spacetime, and give operators a dependance in space and time. Doing so, we get field of operators, where for exemple given a the creation operator field, $a(x,t)$ is an operator creating a particle at x at time t. With this, we can make the theory Poincaré covariant, and in general covariant with any symmetry group. Let G be a symmetry group acting on the spacetime, and U its action on the Fock space. We do have a natural action of G on the Fock space, as any state in the Fock space describes particles across spacetime, and thus naturally transform under any member of G acting on the spacetime. The theory has this symmetry if and only if for any $g \in G$, for any field of operator φ, we have

$$U(g)\varphi(x,t)U(g)^{-1} = \varphi(g(x,t))$$

We will add one last refinement to the quantum mechanics framework before formally defining quantum field theory. Right now, we are considering fields of operator. Let $|\phi\rangle$ the state of a system. Suppose we want to find the momentum of a particle at x at time t. Actually, due to the principle of quantum mechanics, the probability of finding this particle exactly at x at time t is null. With P the momentum operator field, we should have $\langle\phi|P(x,t)|\phi\rangle = 0$. We see that considering fields of operator is not a good approach. The right approach would be to consider operator-valued distributions on the spacetime instead, and to compute $\langle\phi|P(f)|\phi\rangle = 0$ where f is a test function approaching $\delta_{(x,t)}$.

1.1.2 Quantum field theory

With all of these refinements, we can define a quantum field theory, according to Wightman's axioms, which are widely used axioms. Let M be the d dimensional Minkowski space. An Hilbert space $\mathcal{H}$ and a collection (ϕ_a) of operator-valued distributions on M make a *quantum field theory* if

1. We have a unitary representation $U : (q,\Lambda) \to U(q,\Lambda)$ of the Poincaré group such that for all $q \in M, \Lambda \in L$ the Poincaré group, for all a, f, we have

$$U(q,\Lambda)\phi_a(f)U(q,\Lambda)^{-1} = \phi_a((q,\Lambda)f) \tag{1.4}$$

2. We have a vacuum vector $|0\rangle$ fixed by U, and in the domain of any polynomial having as indeterminates the (ϕ_a)
3. Writing for $q \in M$ $U(q,1) = e^{i\sum_{k=0}^{d-1} q_k P_k}$, the joint spectrum of the P_k lies in the forward cone
4. The linear subspace of the polynomials having as indeterminates the (ϕ_a) applied to $|0\rangle$ is dense in $\mathcal{H}$: $\{P|0\rangle / P \in \mathbb{P}((\phi_a))\}$ is dense in $\mathcal{H}$
5. For any test functions f, g whose supports are spacelike separated (any two point is spacelike separated), for any a, b, we have $\phi_a(f)\phi_b(g) = \phi_b(g)\phi_a(f)$

The first axiom makes sure our theory is Poincaré covariant. The second axiom gives a stable vacuum vector. The third makes sure that we have every field we need and that our space is complete. The last one is a locality condition, and makes sure two particles have no interaction if they are spacelike separated.

1.1.3 Conformal field theory

A major difference between quantum mechanics and quantum field theory is that quantum field theory is explicitely Poincaré covariant. More specifically, the first axiom (1.4) makes sure that the Hilbert space contains a representation of the Poincaré group, and that the fields are covariant with its action. But one can extend the symmetry group of the theory, and make it invariant through other kinds of transformations.

Let's first add a few definitions.

Definition 1.1.1. A coordinate transformation is called *conformal* if it preserves angle, or equivalently if it is a local rescaling of the metric. Mathematically speaking, g is a conformal transformation if under a coordinate transformation $x^\mu \to w^\mu(x)$ the spacetime metric transforms as

$$g'_{\mu\nu}(w) = \Lambda(x) g_{\mu\nu}(x) \tag{1.5}$$

Definition 1.1.2. A conformal transformation is said to be *global* if it is invertible.

We call a conformal field theory (or a CFT) any quantum field theory invariant through global conformal transformations. In term of axioms, it is equivalent to extending the unitary representation of the Poincaré group to the global conformal group, and to ensure the fields are covariant under global conformal transformations.

1.2 2D Conformal field theories

1.2.1 2-dimensional conformal transformations

2-dimensional conformal transformations

Conformal transformations become especially interesting when we are working with a dimension of 2. The idea behind CFTs is simply to add symmetries.

Each symmetry adds a constraint on our theory; so adding more symmetries allow us to do more precise models, with less possibilities. It often allows us for more detailed calculations. And in 2 dimensions, the group of conformal transformations is infinite dimensional!

Let's try to actually find these transformations. We take a 2 dimensional space with the Euclidian metric. With Einstein's notations, for a coordinate transformation $x^\mu \to w^\mu(x)$, the metric tensors transforms as

$$g'^{\mu\nu} = \left(\frac{\partial w^\mu}{\partial x^\alpha}\right)\left(\frac{\partial w^\nu}{\partial x^\beta}\right) g^{\alpha\beta} \tag{1.6}$$

For it to be a conformal transformation, for all $\mu, \nu \in \{0, 1\}$, we must have

$$g'^{\mu\nu} = \Lambda g^{\mu\nu} \tag{1.7}$$

With Λ depending on the position. g is the Euclidian metric, so we have $g^{01} = g^{10} = 0$ and $g^{11} = g^{00} = 1$. The condition for the transformation to be conformal thus becomes

$$g'^{00} = g'^{11} = 0 \tag{1.8}$$

$$g'^{01} = g'^{10} \tag{1.9}$$

Adding (1.6) to the above equations, we get the following two conditions:

$$\frac{\partial w^0}{\partial x^0}\frac{\partial w^1}{\partial x^0} + \frac{\partial w^0}{\partial x^1}\frac{\partial w^1}{\partial x^1} = 0 \tag{1.10}$$

$$\left(\frac{\partial w^0}{\partial x^0}\right)^2 + \left(\frac{\partial w^0}{\partial x^1}\right)^2 = \left(\frac{\partial w^1}{\partial x^0}\right)^2 + \left(\frac{\partial w^1}{\partial x^1}\right)^2 \tag{1.11}$$

Writing $\partial_0 \equiv \frac{\partial}{\partial x^0}$ and $\partial_1 \equiv \frac{\partial}{\partial x^1}$, the above conditions can be resumed to

$$\partial_0 w_1 = \pm \partial_1 w_0 \qquad \partial_0 w_0 = \mp \partial_1 w_1 \tag{1.12}$$

These exactly corresponds to the holomorphic and anti-holomorphic Cauchy-Riemann equations. From this, we use the Wick rotation on the plane, to easily separate the holomorphic and anti-holomorphic part of each function. We define

$$\begin{aligned} z &\equiv x^0 + \mathrm{i}x^1 \quad & \partial &\equiv \frac{1}{2}(\partial_0 - \mathrm{i}\partial_1) \\ \bar{z} &\equiv x^0 - \mathrm{i}x^1 \quad & \bar{\partial} &\equiv \frac{1}{2}(\partial_0 + \mathrm{i}\partial_1) \end{aligned} \tag{1.13}$$

Note that the metric tensor thus changes to

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \tag{1.14}$$

With these coordinates, the holomorphic Cauchy-Riemman equations become

$$\bar{\partial} w(z, \bar{z}) = 0 \tag{1.15}$$

Which solution is any holomorphic function $z \to w(z)$ We get the same thing for the antiholomorphic Cauchy-Riemman equations, in the variable $\bar{z}$. But we know that the solution to these equations are functions holomorphic in some open set. The conformal group for $d = 2$ is thus the set of analytical functions, on some open subset. It has an infinite number of dimensions, corresponding to the coefficients of a Laurent serie.

z and $\bar{z}$ are interchangeable everywhere, and independent from each other in the equations as we have seen so far. As such, they are uncorrelated and a 2D CFT is composed of the sum of two algebras, one in the z dimension and the other in the $\bar{z}$ dimension. The data of only one of the two algebras is called a *2D chiral conformal field theory*. As the two chiral algebras are uncorrelated, we will now do most calculation according to z only.

Global conformal transformations

To form a real group, every element must be invertible. Let f be a global conformal transformation on a two-dimensional space, that is an invertible conformal transformation. We have seen previously f is analytical. It must be invertible so injective: f can't have essential singularities nor branch points. Thus, there exist $P, Q \in \mathbb{C}[X]$ such that

$$f(z) = \frac{P(z)}{Q(z)} \tag{1.16}$$

If P has multiple roots, f is not injective. The same goes if Q has multiple roots. As such, there exist $a, b, c, d \in \mathbb{C}$ such that

$$f(z) = \frac{az + b}{cz + d} \tag{1.17}$$

For f to be invertible, the determinant $ad - bc$ must be different than 0. As the choice of a, b, c, d is not unique in (1.17), we can normalize and choose $ad - bc = 1$. Reciprocally, we can easily verify that for any $a, b, c, d, e, f, g, h \in \mathbb{C}$ such that $ad - bc = 1, eh - fg = 1$, the transformation $z, \bar{z} \to \frac{az+b}{cz+d}, \frac{e\bar{z}+f}{g\bar{z}+h}$ is a global conformal transformation.

We conclude that the group of global conformal transformation of dimension 2 is isomorphic to $SL(2, \mathbb{C})^2$.

1.2.2 The Virasoro algebra

This section is mainly inspired by [Jer21]. We now consider an infinitesimal transformation $\epsilon(z)$. We have proved that an infinitesimal conformal transformation in two dimensions must obey the holomorphic Cauchy-Riemman equations, and thus be holomorphic on some open subset. However, our infinitesimal transformation can have singularities outside of the open subset. We will therefore assume that ϵ is meromorphic, and write its Laurent serie

$$z' = z + \epsilon(z) = z - \sum_{n \in \mathbb{Z}} \epsilon_n z^{n+1} \tag{1.18}$$

Remark. Remember that we are only looking at the holomorphic part. The same thing is valid for the anti-holomorphic dimension.

Let's try to find the generator of the transformation made by the n^{th} term. Applying the transformation on a spinless, dimensionless field $\phi(z)$ gives $\phi'(z) =$

$\phi(z)+\delta\phi(z) = \phi(z)+\epsilon(z)\partial\phi(z)$. Thus, the generator associated to the n^{th} term of the serie is

$$l_n = -z^{n+1}\partial \tag{1.19}$$

We thus have infinitely many independent infinitesimal conformal transformations for $d = 2$. If we try to find the conformal algebra, we have

$$\begin{aligned}[l_m, l_n] &= -z^{m+1}\partial(-z^n + 1)\partial + z^{n+1}\partial(-z^m + 1)\partial \\ &= (n-m)z^{m+1+n}\partial \\ &= (m-n)l_{m+n}\end{aligned} \tag{1.20}$$

These bracket relations define what is called a *Witt algebra*. We also have as expected

$$[\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n} \tag{1.21}$$

$$[l_m, \bar{l}_n] = 0 \tag{1.22}$$

However, mathematicians didn't stop their studies here, satisfied by this algebra. Thanks to Lie algebra theory, we know that this algebra is extensible, with what is known as a central extension. Moreover, the extended algebra would be equivalent to the original one, and is preferable to study. Adding a central extension to the Witt algebra modifies (1.20) as following, for $c \in \mathbb{C}$ and some function g

$$[l_m, l_n] = (m-n)l_{m+n} + cg(m,n) \tag{1.23}$$

(1.22) is modified similarly.

The next step is to find what is g. First, it is antisymmetric. We can add $\frac{cg(-1,1)}{2}$ to l_0 such that we now have $g(-1,1) = 0$. Similarly, we can add a constant times $g(n,0)$ to l_n such that for all $n \neq 0, g(n,0) = 0$. We only add constants, so the algebra itself isn't modified.

We are working with the extension of a Lie algebra. The (l_n) still verify the Jacobi identity $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$. Using l_n, l_m and l_0, we then prove that for any m, n such that $m \neq -n$, we have $g(n,m) = 0$. Moreover, thanks to an induction, we prove with the help of l_n, l_0 and l_{n-1} that for any n we have $g(n,-n) = \frac{(n+1)!}{3!(n-2)!}p(2,-2)$. Choosing by convention $p(2,-2) = \frac{1}{2}$ for nice scaling dimension values, we have $g(n,-n) = \frac{1}{12}(n^3 - n)$. More details of this proof can be found here [Jer21, p. 7]. The Lie algebra we then get is called the Virasoro algebra.

Definition 1.2.1. We call a *Virasoro algebra* a Lie algebra composed of $(l_{n\in\mathbb{Z}})$ defined by the following relation for all $n, m \in \mathbb{Z}$

$$[l_m, l_n] = (m-n)l_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m=-n} \tag{1.24}$$

We then call c its *central charge*.

Every 2D CFT contains 2 representations of the Virasoro algebra: one in the holomorphic dimension and the other one in the anti-holomorphic dimension. The Virasoro algebra is the algebra corresponding to conformal transformations. The first axiom of Wightman's axioms (1.4) adapted to conformal field theories is therefore equivalent to saying that every field must be covariant under the action of the 2 Virasoro algebras.

1.2.3 Primary and quasi-primary fields

We take the coordinates from the previous subsection. Let $\phi(z, \bar{z})$ be a field of a 2D CFT.

Definition 1.2.2. We say ϕ is *holomorphic* (or *chiral*) if it only depends on z, and we say it is *anti-holomorphic* (or *anti-chiral*) if it only depends on $\bar{z}$

A CFT is invariant through the conformal group, so through rescaling too. In particular, under the rescaling $z \to \lambda z$, a field changes as

$$\phi(z, \bar{z}) \to \lambda^h \bar{\lambda}^{\bar{h}} \phi(\lambda z, \bar{\lambda}\bar{z}) \tag{1.25}$$

We say h is its holomorphic scaling dimension, and $\bar{h}$ its anti-holomorphic scaling dimension.

Definition 1.2.3. A field ϕ is said to be *quasi-primary* if for any global conformal transformation $z \to f(z)$, the field transforms as

$$\phi(z, \bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial f}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z, \bar{z})) \tag{1.26}$$

Definition 1.2.4. A field ϕ is said to be *primary* if for any conformal transformation $z \to f(z)$, the field transforms as described by equation (1.26).

1.2.4 The Hilbert space

We want to see what the Hilbert space of a conformal theory looks like. We will simply give an overview without going into much details. More can be found here [FMS97, p. 200] We consider a 2D conformal field theory. We write $(L_n)_n$ its Virasoro algebra in the holomorphic dimension, and $(\bar{L}_n)_n$ its Virasoro algebra in the antiholomorphic dimension. L_{-1}, L_0, L_1 and their antiholomorphic counterparts generates global conformal symmetries. We thus want to define the vacuum state $|0\rangle$ such that

$$\begin{aligned} L_{-1}|0\rangle &= 0 \qquad & L_0|0\rangle &= 0 \qquad & L_1|0\rangle &= 0 \\ \bar{L}_{-1}|0\rangle &= 0 \qquad & \bar{L}_0|0\rangle &= 0 \qquad & \bar{L}_1|0\rangle &= 0 \end{aligned} \tag{1.27}$$

More generally, we want for any $n \geq -1$

$$\begin{aligned} L_n|0\rangle &= 0 \\ \bar{L}_n|0\rangle &= 0 \end{aligned} \tag{1.28}$$

From this, we proceed as for the usual construction of a representation of $su(2)$. As the two Virasoro algebras are decoupled, the space generated by the two Virasoro algebras will simply be the tensor product of the two spaces each generated by a Virasoro algebra. For this reason, we will for now only consider the holomorphic dimension.

We want to find the space of state spanned from a highest-weight state by the holomorphic Virasoro algebra. We can define a highest-weight state $|h\rangle$, eigenstate of L_0 with eigenvalue h. Such a state is created by applying asympotically a primary field ϕ_h of holomorphic dimension h to the vacuum

$$|h\rangle = \phi_h(0)|0\rangle \tag{1.29}$$

Remark. We can choose to take the field at 0 due to conformal invariance, where we can map any non-dense set of points to a neighboorhood of 0. Currently, choosing to take the fields in 0 doesn't make much sense. This will make more sense in 3.1.

Since $[L_0, L_m] = -mL_m$, the $(L_m)_{m>0}$ act as lowering operators on $|h\rangle$, whilst the $(L_{-m})_{m>0}$ act as raising operators. We have for $m > 0$

$$L_{-m}|h\rangle = 0 \tag{1.30}$$

All other states are generated by applying any combination of lowering operators on $|h\rangle$. A general state takes the form

$$L_{-k_1} \dots L_{-k_n}|h\rangle \tag{1.31}$$

where by convention we have $k_1 \leq \cdots \leq k_n$. Note that we thus have an equivalence between states and operators: to any state, we can associate a unique operator field (made of the primary field to which we applied Virasoro operators) which, when applied to the vacuum, gives the state.

We call this space of states a *Verma module*, and write it $V(c, h)$ where c is the central charge of the Virasoro algebra. Then, in general, the Hilbert space of the theory takes the form

$$\mathcal{H} = \sum_{h,\bar{h}} V(c, h) \otimes \bar{V}(c, \bar{h}) \tag{1.32}$$

Remark. Note that in general, a Verma module is not an irreductible representation of the Virasoro algebra. In particular, one may find sub Verma modules inside a Verma module.

1.3 The free boson

Let's now try to put our knowledge to use. One of the simplest system one can consider is the free scalar field φ, a scalar field with action

$$\begin{gathered} S[\varphi] = \int \mathrm{d}x\mathrm{d}t\mathcal{L}(\varphi, \dot{\varphi}, \nabla\varphi) \qquad \dot{\varphi} \equiv \frac{\partial\varphi}{\partial t} \\ \mathcal{L} = \frac{1}{2}\left(\frac{1}{c^2}\dot{\varphi}^2 - (\nabla\varphi)^2 - m^2\varphi^2\right) \end{gathered} \tag{1.33}$$

For the rest of this section, we set $c = 1$. We consider this system in 2 dimensions, one temporal and one spatial dimension. We can start to study the system by replacing the spatial dimension with a discrete set of points. We consider N points, with a lattice spacing of a. Moreover, we set a periodic boundary condition $\varphi_N = \varphi_0$. The Lagrangian (1.33) then becomes

$$\mathcal{L} = \sum_{k=0}^{N-1} \frac{a}{2}\left(\dot{\varphi}_k^2 - \frac{1}{a^2}(\varphi_{k+1} - \varphi_k)^2 - m^2\varphi_k^2\right) \tag{1.34}$$

We define the canonical momentum conjugate to the variable φ_n:

$$\pi_n = a\dot{\varphi}_n \tag{1.35}$$

Rewriting the Lagrangian (1.34) in terms of the position and momentum, we get the Hamiltonian

$$H = \frac{1}{2}\sum_{k=0}^{N-1}\left(\frac{1}{a}\pi_k^2 - \frac{1}{a}(\varphi_{k+1}-\varphi_k)^2 - am^2\varphi_k^2\right) \tag{1.36}$$

We switch to the canonical quantization, by replacing the (φ_k) and (π_k) by operators, and by imposing at equal times the following commutation relations:

$$\begin{aligned}[\varphi_n, \pi_m] &= i\delta_{n,m} \\ [\pi_n, \pi_m] &= [\varphi_n, \varphi_m] = 0\end{aligned} \tag{1.37}$$

We have set Planck's constant to 1, for simplicity.

The Hamiltonian is translation invariant. This motivates the use of Fourier transforms.

$$\begin{aligned}\tilde{\varphi}_k &= \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} e^{-2\pi ikj/N}\varphi_j \\ \tilde{\pi}_k &= \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} e^{-2\pi ikj/N}\pi_j\end{aligned} \tag{1.38}$$

φ_k and π_k are real so $\tilde{\varphi}_k^\dagger = \tilde{\varphi}_{-k}$ and $\tilde{\pi}_k^\dagger = \tilde{\pi}_{-k}$. We have $[\tilde{\varphi}_n, \tilde{\pi}_m^\dagger] = i\delta_{n,m}$

With these, the Hamiltonian (1.36) becomes

$$H = \frac{1}{2}\sum_{k=0}^{N-1}\left(\frac{1}{a}\tilde{\pi}_k\tilde{\pi}_k^\dagger + a\tilde{\varphi}_k\tilde{\varphi}_k^\dagger\left[m^2 + \frac{2}{a^2}\left(1 - cos(\frac{2\pi k}{N})\right)\right]\right) \tag{1.39}$$

This is exactly the Hamiltonian for a system of uncoupled harmonic oscillators, with frequencies

$$\omega_k = \sqrt{m^2 + \frac{2}{a^2}\left(1 - cos(\frac{2\pi k}{N})\right)} \tag{1.40}$$

We can then define the annihilation and creation operators

$$\begin{aligned}a_k &= \frac{1}{\sqrt{2a\omega_k}}(a\omega_k\tilde{\varphi}_k + i\tilde{\pi}_k) \\ a_k^\dagger &= \frac{1}{\sqrt{2a\omega_k}}(a\omega_k\tilde{\varphi}_k^\dagger - i\tilde{\pi}_k^\dagger)\end{aligned} \tag{1.41}$$

We do have

$$[a_n, a_m^\dagger] = \delta_{n,m} \tag{1.42}$$

The Hamiltonian rewrites nicely as

$$H = \sum_{k=0}^{N-1}(a_k^\dagger a_k + \frac{1}{2})\omega_k \tag{1.43}$$

We see there that a_k and $a_k^\dagger$ indeed act as annihilation and creation operators, with $a_k^\dagger a_k$ the density operator. We define the ground state $|0\rangle$ such that for all k, $a_k|0\rangle = 0$. From this, any state can be computed by repeatedly applying creation operators on the ground state.

From this, we can let $a \to 0$ and $N \to \infty$ simultaneously, such that Na stays constant. This way, we get back to the continuous system in 2 dimensions. φ_k becomes $\varphi(x)$, $\frac{1}{a}\pi_n$ becomes $\pi(x) = \dot{\varphi}(x)$. $a\sum_{k=0}^{N-1}$ becomes $\int \mathrm{d}x$, and $\delta_{n,m}$ becomes $a\delta(x - x')$.

Similarly, the Fourier indices k are replaced by the momentum $p = \frac{2\pi k}{V}$. $\sum_{k=0}^{N-1}$ becomes $\frac{V}{2\pi}\int \mathrm{d}p$, and a_k is replaced by $\frac{1}{V}a(p)$, with energy $\omega(p) = \sqrt{m^2 + p^2}$.

The simplest states are of the form $a^\dagger(p)|0\rangle$, with energy $\omega(p) = \sqrt{m^2 + p^2}$. This is typical of relativistic particles. We can thus interpret these states as elementary particles. Moreover, general states aren't affected by switching two particles ($a^\dagger(p)a^\dagger(p')|0\rangle = a^\dagger(p')a^\dagger(p)|0\rangle$). We are thus considering bosons. There is no interactions between the bosons, so we call this system the free boson system. As of now, the system is not particularly conformal invariant. In particular, a unit of mass is introduced in the system due to the mass of the boson taken into accoun in the Lagrangian. To make this theory conformally invariant, we need to consider the free massless boson. This is what we will do in the next section on the boson.

Chapter 2

Correlation functions

Now that we have defined conformal field theories, we would like to study their dynamics, and how conformal invariance actually constrains the theory. In 2.1, we define correlation functions and see how they are constrained by conformal invariance. In 2.2, we use the symmetries of our system to derive a family of equations constraining the theory. In particular, we define the energy-momentum tensor. Finally, in 2.3, we compute the correlation functions in the theory of the free boson.

2.1 Correlation functions

2.1.1 Motivation

The main objective of quantum field theory, and thus conformal field theory, is to compute correlation functions. Let's first try to understand how it works in classical mechanics.

Suppose we are in presence of a potential V, vanishing at infinity. Let (x, v) be the position and velocity of a particle at time 0. Without potential, this particle would be in $(x - tv, v)$ at time $-t$. We write $\Omega_t^-(x, v) = (x - tv, v)$. Reciprocally, knowing the position and velocity of a particle (x, v) at time 0, we can compute $\Omega_t^+(x, v)$ the position and velocity of the same particle under the potential V at time t. We then have $\lim_{t\to\infty} \Omega_t^+\Omega_t^-(x, v)$ the position and velocity of the particle coming from the "direction (x, v)", under the action of the potential V. Finally, we can compute $\lim_{t\to\infty} \Omega_t^-\Omega_t^+\Omega_t^+\Omega_t^-(x, v)$, the direction in which the particle ends when it came from the direction (x, v). We call $S = \lim_{t\to\infty} \Omega_t^-\Omega_t^+\Omega_t^+\Omega_t^-$ the scattering operator. Understanding this operator allows one to compute the asymptotic states of the system, which is often what we desire.

By analogy, we can also use similar methods in quantum mechanics. With H the Hamiltonian of a system and H_0 the free Hamiltonian (the Hamiltonian of the system without external forces), we can define the evolution and free evolution operators. We have $\Omega_t^- = e^{-itH_0}$ and $\Omega_t^+ = e^{itH}$. We can then define the scattering operator (under some conditions) and compute the probability of a particle coming from a direction ending in another direction

$$\mathbb{P}((x,v) \to (x',v')) = \langle x',v'|S|x,v\rangle \tag{2.1}$$

Using the quantum field theory formalism, we can go even further. Using the creation a and annihilation $a^\dagger$ operator fields, we can write

$$\mathbb{P}((x,v) \to (x',v')) = \langle 0|a^\dagger(x',v')Sa(x,v)|0\rangle \tag{2.2}$$

In a more general way, we want to compute expressions of the form

$$\langle 0|\varphi_1(x_1)\varphi_2(x_2)\dots\varphi_n(x_n)|0\rangle \tag{2.3}$$

where $(\varphi_k)_k$ is a sequence of fields. As we want this to express the probability of a state becoming another one through time, we want the operators inside of the expression to be ordered according to time, meaning x_1 is later than x_2, which is later than x_3, etc... This leads to the definition of a correlation function.

Definition 2.1.1. Given a sequence of fields, the *n-point correlation function*, or simply correlation function, of a sequence of field $(\varphi_k)_k$, is the function in n variables

$$\langle 0|\mathcal{T}(\varphi_1(x_1)\varphi_2(x_2)\dots\varphi_n(x_n))|0\rangle \tag{2.4}$$

where $\mathcal{T}$ is the time-ordering operator, which reorders the fields according to the time they are taken at.

Remark. Note that the correlation function is not a real correlation function, in the sense of statistics or probability. In fact, the correlation function of two fields can be negative. However, it is the most correct generalisation of what we could imagine as a correlation function between fields.

2.1.2 Path integral formalism

We would like to use the conformal symmetry of conformal field theories to add constraints to correlation functions, to compute them more easily. To do so, we need to know how a correlation function transforms along with a coordinate transformation. The best way to see how correlation functions is to switch to the path integral formalism.

We will simply see here the main idea of the path integral formalism. A more detailed discussion can be found here [Fra].

The main idea is to consider the action of the system. We usually consider the Hamiltonian, due to one of the axioms of quantum mechanics which directly gives an Hamiltonian. However, one could also consider the Lagrangian $\mathcal{L}$ and the action $\mathcal{S} = \int \mathcal{L}$. Well defining the action, for 2 fields φ_1, φ_2 taken in a point x and in two times t_1, t_2, we have

$$\langle \varphi_1(x,t_1)|\varphi_2(x,t_2)\rangle = \int [d\varphi(x,t)]e^{i\mathcal{S}[\varphi]} \tag{2.5}$$

where φ is an interpolation between the fields φ_1 and φ_2, where $[d\varphi(x,t)]$ is a measure on spacetime depending on φ, and where $\mathcal{S}$ is the action of the system.

As a matter of fact, the action of the system is often defined for (2.5) to be true. Let's now try to reformulate correlation functions using the action of the system.

Let ψ an arbitrary field. Writing $|n\rangle$ the energy eigenstates of the Hamiltonian with eigenvalue E_n, we have

$$\begin{aligned} e^{itH(1-i\epsilon)}|\psi\rangle &= \sum_n e^{itH(1-i\epsilon)}|n\rangle\langle n|\psi\rangle \\ &= \sum_n e^{itE_n(1-i\epsilon)}|n\rangle\langle n| \\ &\to_{\epsilon\to 0, t\to\infty} e^{itE_0(1-i\epsilon)}|0\rangle\langle 0|\psi\rangle \end{aligned} \tag{2.6}$$

Then for ψ_1, ψ_2 two arbitrary fields, for any two operators $\mathcal{O}_1, \mathcal{O}_2$, we have

$$\frac{\langle 0|\mathcal{O}_1|0\rangle}{\langle 0|\mathcal{O}_2|0\rangle} = \lim_{t_1,t_2\to\infty,\epsilon\to 0} \frac{\langle\psi_1|e^{-it_1H(1-i\epsilon)}\mathcal{O}_1 e^{-it_2H(1-i\epsilon)}|\psi_2\rangle}{\langle\psi_1|e^{-it_1H(1-i\epsilon)}\mathcal{O}_2 e^{-it_2H(1-i\epsilon)}|\psi_2\rangle} \tag{2.7}$$

Thus, for a sequence of fields $(x_k)_k$ and a sequence of times $(t_k)_k$, recalling that $x_k(t_k) = e^{it_kH}x_k e^{-it_kH}$ with $x_k = x_k(0)$ (1.2), assuming that the $(x_k)_k$ and $(t_k)_k$ are already time-ordered, we have

$$\begin{aligned} \langle x_1(t_1)\ldots x_n(t_n)\rangle &= \frac{\langle 0|x_1 e^{iH(t_2-t_1)}x_2 e^{iH(t_3-t_2)}\ldots e^{iH(t_n-t_{n-1})}x_n|0\rangle}{\langle 0|e^{iH(t_n-t_1)}|0\rangle} \\ &= \frac{\langle\psi_1|e^{-iT_1H(1-i\epsilon)}x_1 e^{iH(t_2-t_1)}\ldots x_n e^{-iT_2H(1-i\epsilon)}|\psi_2\rangle}{\langle\psi_1|e^{-i(T_1+T_2+t_1-t_n)H(1-i\epsilon)}|\psi_2\rangle} \end{aligned} \tag{2.8}$$

for $T_1, T_2 \to \infty, \epsilon \to 0$. By inserting sums of $|n\rangle\langle n|$ between each operator and by replacing everything in the path formalism, we obtain for the nominator

$$\lim_{T_1,T_2\to\infty,\epsilon\to 0} \int_{T_1}^{T_2} [dx(t)]\psi_1^*(T_1)\psi_2(T_2)x_1(t_1)\ldots x_n(t_n)e^{iS_\epsilon[x(t)]} \tag{2.9}$$

Remembering that the fields ψ_1, psi_2 where chosen arbitrarily, we can choose them such that $\psi_1(T_1) = \psi_2(T_2) = 1$. We have

$$\langle x_1(t_1)\ldots x_n(t_n)\rangle = \lim_{\epsilon\to 0} \frac{\int [dx(t)]x_1(t_1)\ldots x_n(t_n)e^{iS_\epsilon[x(t)]}}{\int [dx(t)]e^{iS_\epsilon[x(t)]}} \tag{2.10}$$

With the change of coordinates $t \to -i\tau$, redefining $x_k(-i\tau)$ as $x_k(\tau)$, we finally have

$$\langle x_1(\tau_1)\ldots x_n(\tau_n)\rangle = \frac{\int [dx]x_1(\tau_1)\ldots x_n(\tau_n)e^{-S_E[x(\tau)]}}{\int [dx]e^{-S_E[x(\tau)]}} \tag{2.11}$$

with $S_E(x(\tau)) = -iS(x(t))$.

Remark. This transformation leads to what is called the Euclidian formalism. To do the transformation, we have assumed that the correlation functions could be analytically continued from real time to imaginary time.

2.1.3 Conformal invariance

Now that we have an expression of correlation functions in the path integral formalism, we can easily see how they transform. Let S be the action of the system in the Euclidian formalism. Let $(\varphi_k)_k$ a sequence of fields, and $(x_k)_k$ a sequence of coordinates. We write $Z = \int [d\varphi] e^{-S[\varphi]}$ the vacuum functional. By analogy with statistical mechanics, we sometimes also call Z the partition function. Let $x_k \to x'_k$ a coordinate transformation under which the action is invariant. We write

$$\varphi'(x') = \mathcal{F}(\varphi(x)) \tag{2.12}$$

We have

$$\begin{aligned} \langle \varphi_1(x'_1) \dots \varphi_n(x'_n) \rangle &= \frac{1}{Z} \int d[\varphi] \varphi_1(x'_1) \dots \varphi_n(x'_n) e^{-S[\varphi]} \\ &= \frac{1}{Z} \int d[\varphi'] \varphi'_1(x'_1) \dots \varphi'_n(x'_n) e^{-S[\varphi']} \\ &= \frac{1}{Z} \int d[\varphi] \mathcal{F}(\varphi_1(x_1)) \dots \mathcal{F}(\varphi_n(x_n)) e^{-S[\varphi]} \\ &= \langle \mathcal{F}(\varphi_1(x_1)) \dots \mathcal{F}(\varphi_n(x_n)) \rangle \end{aligned} \tag{2.13}$$

In particular, a conformal field theory is a quantum field theory where the axiom is invariant under conformal transformations. Recalling (1.25), for any sequence of field $(\phi_k)_k$ in a conformal field theory with scaling dimensions $(\Delta_k)_k$, for any sequence of points $(x_k)_k$, for $\lambda \in \mathbb{R}$ we have

$$\langle \phi_1(\lambda x_1) \dots \phi_n(\lambda x_n) \rangle = \lambda^{-\Delta_1 \dots - \Delta_n} \langle \phi_1(x_1) \dots \phi_n(x_n) \rangle \tag{2.14}$$

Now consider a 2 dimensional conformal field theory. Let $(\phi_k)_k$ be a sequence of primary fields, with conformal dimensions $(h_k)_k$ and $(\bar{h}_k)_k$. According to (1.26), for a conformal transformation of the form $z \to w, \bar{z} \to \bar{w}$,(2.13) becomes

$$\begin{aligned} &\langle \phi_1(w_1, \bar{w}_1) \dots \phi_n(w_n, \bar{w}_n) \rangle \\ &\quad = \prod_{k=1}^{n} \left(\frac{\mathrm{d}w}{\mathrm{d}z} \right)_{w=w_k}^{-h_k} \left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}} \right)_{\bar{w}=\bar{w}_k}^{-\bar{h}_k} \langle \phi_1(z_1, \bar{z}_1) \dots \phi_n(z_n, \bar{z}_n) \rangle \end{aligned} \tag{2.15}$$

Let's now look at the 2-points correlation function of primary fields in a 2D CFT. Due to rotation and translation invariance, we have in any dimension that

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = f(|x_1 - x_2|) \tag{2.16}$$

In 2 dimensions, we are using the coordinates defined in (1.13). $|x_1, x_2|$ becomes $((z_1 - z_2)(\bar{z}_1 - \bar{z}_2))^{\frac{1}{2}}$.

The above equation becomes:

$$\langle \phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2) \rangle = f((z_1 - z_2)(\bar{z}_1 - \bar{z}_2)) \tag{2.17}$$

But with (2.14), this is constrained to

$$\langle \phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2) \rangle = \frac{C}{(z_1 - z_2)^{h_1 + h_2} (\bar{z}_1 - \bar{z}_2)^{\bar{h}_1 + \bar{h}_2}} \tag{2.18}$$

with C a constant depending on the 2 fields. Finally, using the covariance of the correlation function with special conformal transformations, we get an equation implying that the conformal dimensions of the two fields must be equal [FMS97, p. 105]. If the conformal dimensions of the two fields are different, the correlation function necessarily vanishes. Else, we have

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\rangle = \frac{C}{(z_1 - z_2)^{2h}(\bar{z}_1 - \bar{z}_2)^{2\bar{h}}} \tag{2.19}$$

with $h = h_1 = h_2$, $\bar{h} = \bar{h_1} = \bar{h_2}$.

2.1.4 Wick's theorem

We will finish this section with a theorem coming from combinatorics, which will prove very useful later. The demonstration of this theorem is not too hard but really tedious. We will not go through it, but it can be found here [Eva18].

Let's first define the normal ordering.

Definition 2.1.2. The *normal ordering* of a sequence of operators $\phi_1 \dots \phi_n$, written $:\phi_1 \dots \phi_n:$, is the product of these operators where any annihilation operator has been put to the right, and any creation operator has been put to the left.

For exemple, given a theory with a creation operator $a^\dagger$ and a annihilation operator a, $: aa^\dagger : = a^\dagger a$. This ordering prevents one from computing the vacuum's energy expectation, which often diverges.

Definition 2.1.3. Given a normal-ordered product of operators $:\phi_1 \dots \phi_n:$, we define the *contraction* of the 2 operators ϕ_i and ϕ_j as the normal-ordered product $:\phi_1 \dots \phi_n:$ where we removed ϕ_i and ϕ_j, multiplied by their correlation function $\langle \phi_i \phi_j \rangle$. We write it

$$:\wick{\phi_1 \dots \c\phi_i \dots \c\phi_j \dots \phi_n}:$$

For exemple,

$$\begin{aligned} :\wick{\phi_1 \c\phi_2 \phi_3 \c\phi_4}: &=: \phi_1\phi_3 : \langle \phi_2\phi_4 \rangle \\ :\wick{\c1\phi_1 \c2\phi_2 \c1\phi_3 \c2\phi_4}: &= \langle \phi_1\phi_3 \rangle \langle \phi_2\phi_4 \rangle \end{aligned} \tag{2.20}$$

Theorem 2.1.1 (Wick's theorem)**.** *The time-ordered product is equal to the normal ordered product, plus all possible ways of contracting pairs within it.*

For exemple,

$$\begin{aligned} \mathcal{T}(\phi_1\phi_2\phi_3\phi_4) = \quad & :\phi_1\phi_2\phi_3\phi_4: + \\ & :\wick{\c\phi_1 \c\phi_2 \phi_3\phi_4}: + :\wick{\phi_1 \c\phi_2 \c\phi_3 \phi_4}: + :\wick{\phi_1\phi_2 \c\phi_3 \c\phi_4}: + \\ & :\wick{\c\phi_1 \phi_2 \c\phi_3 \phi_4}: + :\wick{\phi_1 \c\phi_2 \phi_3 \c\phi_4}: + :\wick{\c\phi_1 \phi_2\phi_3 \c\phi_4}: + \\ & :\wick{\c1\phi_1 \c1\phi_2 \c2\phi_3 \c2\phi_4}: + :\wick{\c1\phi_1 \c2\phi_2 \c2\phi_3 \c1\phi_4}: + :\wick{\c1\phi_1 \c2\phi_2 \c1\phi_3 \c2\phi_4}: \end{aligned} \tag{2.21}$$

2.2 Ward identities

Ward identities are a very important family of identities, derived from the symmetries of a theory. They reflect the constraints put on a theory due to its symmetries.

2.2.1 Transformation generators

Let's first consider an infinitesimal transformation $x^\mu \to x'^\mu$. We take again the notations of 2.12. We can in general parametrize such a transformation by a set of infinitesimal parameters $(\omega_a)_a$ such that we have at first order

$$\begin{aligned} x'^\mu &= x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a} \\ \phi'(x') &= \phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x) \end{aligned} \tag{2.22}$$

Definition 2.2.1. We define the generator G_a of a transformation by the following expression

$$\phi'(x) - \phi(x) \equiv -i\omega_a G_a \phi(x) \tag{2.23}$$

With (2.22), we can write

$$\phi'(x') = \phi(x') - \omega_a \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \phi(x') + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x') \tag{2.24}$$

But according to (2.23),

$$iG_a \phi(x') = \frac{1}{\omega_a}(\phi(x') - \phi'(x')) \tag{2.25}$$

And so

$$iG_a \phi = \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \phi - \frac{\delta \mathcal{F}}{\delta \omega_a} \tag{2.26}$$

To end this subsection, let's try to compute the generators for some symmetries in our system.

Let $x^\mu \to x^\mu + w^\mu = x'^\mu$ be an infinitesimal translation. Here, the index a becomes a spacetime index. We have

$$\frac{\delta x^\mu}{\delta \omega^\nu} = \delta^\mu_\nu \qquad \frac{\delta \mathcal{F}}{\delta \omega^\nu} = 0 \tag{2.27}$$

So the generator of translations is equal to

$$P_\nu = -i\partial_\nu \tag{2.28}$$

Now, let $x^\mu \to \omega^\nu_\mu x^\mu = \omega_{\rho\mu}\eta^{\rho\nu} x^\mu = x'^\mu$ be an infinitesimal Lorentz transformation. ω must be antisymmetric, so we can write

$$\frac{\delta x^\mu}{\delta \omega_{\rho\nu}} = \frac{1}{2}(\eta^{\rho\mu} x^\mu - \eta^{\nu\mu} x^\rho) \tag{2.29}$$

It's effect on the field can be writen $\mathcal{F}(\psi) = L_\omega \psi$. At first order, we can write

$$L_\omega \simeq 1 - \frac{1}{2} i\omega_{\rho\nu} S^{\rho\nu} \tag{2.30}$$

with $S^{\rho\nu}$ an hermitian matrix. Using (2.26), we finally have that the generator of Lorentz transformations is equal to

$$L^{\rho\nu} = i(x^\rho \partial^\nu - x^\nu \partial^\rho) + S^{\rho\nu} \tag{2.31}$$

Finally, let $x^\mu \to (1+\omega)x^\mu$ be a dilatation. According to 1.25, $\mathcal{F}(\phi) = (1+\omega)^{-\Delta}\phi \simeq (1-\omega\Delta)\phi$ with Δ the scaling dimension of the field. We thus have

$$\frac{\delta x^\mu}{\delta \omega^\nu} = x^\mu \delta^\mu_\nu \qquad \frac{\delta \mathcal{F}}{\delta \omega^\nu} = -\Delta \tag{2.32}$$

So the generator of dilatations is equal to

$$D = -ix^\nu \partial_\nu - i\Delta \tag{2.33}$$

2.2.2 General Ward identities

We want to see how an infinitesimal transformation modifies the action of the system, in the path integral formalism. As before, let $x^\mu \to x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a}$ be an infinitesimal transformation, transforming a field as $\phi(x) \to \phi'(x') = \phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x) = \mathcal{F}(\phi(x))$. We have

$$\begin{aligned} S' &= \int d^d x \mathcal{L}(\phi'(x), \partial_\mu \phi'(x)) \\ &= \int d^d x' \mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x')) \\ &= \int d^d x' \mathcal{L}(\mathcal{F}(\phi(x)), \partial'_\mu \mathcal{F}(\phi(x))) \\ &= \int d^d x |\frac{\partial x'}{\partial x}| \mathcal{L}(\mathcal{F}(\phi(x)), \frac{\partial x^\mu}{\partial x'^\nu} \partial_\mu \mathcal{F}(\phi(x))) \end{aligned} \tag{2.34}$$

But at first order, we have

$$\frac{\partial x'^\nu}{\partial x^\mu} \simeq \delta^\nu_\mu + \partial_\mu \left(\omega_a \frac{\delta x^\nu}{\delta \omega_a} \right) \qquad \frac{\partial x^\nu}{\partial x'^\mu} \simeq \delta^\nu_\mu - \partial_\mu \left(\omega_a \frac{\delta x^\nu}{\delta \omega_a} \right) \tag{2.35}$$

Moreover,

$$\det(1+E) \simeq 1 + \mathrm{Tr}(E) \tag{2.36}$$

So

$$|\frac{\partial x'}{\partial x}| \simeq 1 + \partial_\mu \left(\omega_a \frac{\delta x^\mu}{\delta \omega_a} \right) \tag{2.37}$$

Injecting these in (2.34), we get:

$$\begin{aligned} S' = \int d^d x &\left(1 + \partial_\mu \left(\omega_a \frac{\delta x^\mu}{\delta \omega_a} \right) \right) \times \\ &\mathcal{L} \left(\phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x), \left[\delta^\nu_\mu - \partial_\mu \left(\omega_a \frac{\delta x^\nu}{\delta \omega_a} \right) \right] \left(\partial_\nu \phi(x) + \partial_\nu \left[\omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x) \right] \right) \right) \end{aligned} \tag{2.38}$$

We are interested in computing $\delta S = S' - S$. Expanding the Lagrangian to the first order, and defining the *current* associated with the infinitesimal transformation as follows

$$j_a^\mu = \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi - \delta_\nu^\mu \mathcal{L}\right]\frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\frac{\delta \mathcal{F}}{\delta \omega_a} \tag{2.39}$$

we have

$$\delta S = -\int d^d x j_a^\mu \partial_\mu \omega_a \tag{2.40}$$

but, after integrating by parts,

$$\delta S = \int d^d x \partial_\mu j_a^\mu \omega_a \tag{2.41}$$

The main interest of these computations is that according to the laws of motion, δS should vanish for any choice of $\omega_a(x)$. In particular, we can get the following conservation law

$$\partial_\mu j_a^\mu = 0 \tag{2.42}$$

This implies that for any continous symmetry in our theory, we can associated a conserved current.

We can write the conserved charge associated to the current j_a^μ, which is

$$Q_a = \int dx j_a^0 \tag{2.43}$$

Now that we have seen the effect of an infinitesimal transformation on the action, let's see how it affects a correlation function. Let $X = \phi_1(x_1)\dots\phi_n(x_n)$ the product of multiple fields. We write δ_ω its variation under the transformation parameterized by ω. Supposing that the transformation is a symmetry of the theory, the correlation function of X is invariant under the transformation. We can thus rewrite the correlation function formula in the path integral formalism (2.11) after the transformation, and get

$$\langle X\rangle = \frac{1}{Z}\int [d\phi'](X + \delta X)e^{-S[\phi] - \int dx \partial_\mu j_a^\mu \omega_a(x)} \tag{2.44}$$

We suppose as in (2.13) that the measure is invariant under the transformation. Expanding the exponential at the first order, we can compute the variation of the correlation function

$$\langle \delta X\rangle = \int dx \partial_\mu \langle j_a^\mu(x) X\rangle \omega_a(x) \tag{2.45}$$

On the other hand, we can compute the variation of X explicitly using the definition of the transformation generators:

$$\begin{aligned} \delta X &= -i\sum_{k=1}^{n}\left(\phi_1(x_1)\dots G_a\phi_k(x_k)\dots\phi_n(x_n)\right)\omega_a(x_k) \\ &= -i\int dx \omega_a(x)\sum_{k=1}^{n}\left(\phi_1(x_1)\dots G_a\phi_k(x_k)\dots\phi_n(x_n)\right)\delta(x - x_k) \end{aligned} \tag{2.46}$$

But (2.45) and (2.46) are true for any infinitesimal transformation ω_a. We can thus go under the integral, and putting together the two ways to compute $\langle X \rangle$ we get

$$\partial_\mu \langle j_a^\mu(x)\phi_1(x_1)\dots\phi_n(x_n)\rangle = -i\sum_{k=1}^{n}\delta(x-x_k)\langle\phi_1(x_1)\dots G_a\phi_k(x_k)\dots\phi_n(x_n)\rangle \tag{2.47}$$

This relation is called the Ward identity associated with the current j_a^μ.

2.2.3 The energy-momentum tensor

Let's look at the simplest exemples of infinitesimal transformations constituting a symmetry of our system: translations. Recalling what we have previously computed on translations ((2.27)), we can easily compute the current assiociated with translations using the formula (2.39). Changing the index a to ν and moving it up using the metric tensor, we have

$$j^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi \tag{2.48}$$

We name this tensor the *canonical energy-momentum tensor*, and write it $T_c^{\mu\nu} = j^{\mu\nu}$. The name of this tensor comes from the fact that the associated conserved charge obtained with (2.43) is the four-momentum, or energy-momentum

$$P^\nu = \int dx T_c^{0\nu} \tag{2.49}$$

We notice in particular that P^0 corresponds to the usual definition of the Hamiltonian.

Adding the divergence of a tensor $B^{\mu\nu\rho}$ antisymmetric in the first two indices does not affect the conservation of the current. We can thus redefine our current at will by adding the divergence of such a tensor and still verify the associated Ward identity. In particular, thanks to our theory having the Lorentz transformations as symmetry, we can find such a tensor such that the energy-momentum becomes symmetric. We won't go into the details of the computation, which are mostly technical. A demonstration can be found here [FMS97, p. 46]. We usually call the newly defined current the *Belinfante energy-momentum tensor*. Furthermore, in a conformal field theory, we can also add another term to the tensor to make it traceless. The details are once again technical, and can be found here [FMS97, p. 107] for the specific case of dimension 2. We simply call the newly obtained tensor the *energy-momentum tensor*, and write it $T^{\mu\nu}$

We finish this subsection by computing the current associated to dilatations. Injecting what we computed on dilatations (2.32) into the formula for the current (2.39), we have

$$\begin{aligned} j^\mu &= -\mathcal{L}x^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}x^\nu\partial_\nu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi \\ &= T_c{}^\mu{}_\nu x^\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi \end{aligned} \tag{2.50}$$

However, we just saw that we can make the energy-momentum tensor traceless. Moreover, by definition of the current, $\partial_\mu j^\mu = 0$. We have

$$\partial_\mu T^\mu_\nu x^\nu + \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi = 0 \tag{2.51}$$

So

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi = 0 \tag{2.52}$$

We can redefine once again the energy-momentum tensor, and write generally

$$j^\mu = T^\mu_\nu x^\nu \tag{2.53}$$

2.2.4 Ward identities in CFTs

Now that we have defined the energy-momentum tensor, and that we have a convenient way of writing the currents associated with translations and dilatations, let's try to derive the Ward identities associated with the symmetries of our theory.

Let once again $X = \phi_1(x_1)\dots\phi_n(x_n)$. Injecting the energy-momentum tensor and the translation generator (2.28) in the general Ward identity (2.47), we immediatly get

$$\partial_\mu \langle T^\mu_\nu X\rangle = \sum_{k=1}^{n} \delta(x - x_k) \frac{\partial}{\partial x^\nu_k} \langle X\rangle \tag{2.54}$$

This is the Ward identity associated to translations.

Let us now look at Lorentz transformations. Putting (2.29) and (2.30) into the formula for the current (2.39), we get the current associated with Lorentz transformations

$$j^{\mu\nu\rho} = T^{\mu\nu}_c x^\rho - T^{\mu\rho}_c x^\nu + i\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} S^{\nu\rho}\phi \tag{2.55}$$

But by rendering the energy-momentum tensor symmetric, we annihilated the last term, such that

$$j^{\mu\nu\rho} = T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu \tag{2.56}$$

Putting this expression along with the generator for Lorentz transformations defined here (2.31) in the general Ward identity, we get

$$\partial_\mu \langle (T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu) X\rangle = \sum_{k=1}^{n} \delta(x - x_k) \left[(x^\nu_k \partial^\rho_k - x^\rho_k \partial^\nu_k)\langle X\rangle - iS^{\nu\rho}_k \langle X\rangle\right] \tag{2.57}$$

We can develop the divergence on the left hand side using Leibnitz rule. The derivative either acts on the energy-momentum tensor, and can be removed using (2.54), or acts on x^μ and diseappears in a δ^ν_μ. We can thus reduce the expression above, to obtain

$$\langle (T^{\rho\nu} - T^{\nu\rho}) X\rangle = -i \sum_{k=1}^{n} \delta(x - x_k) S^{\nu\rho}_k \langle X\rangle \tag{2.58}$$

This is the Ward identity associated with Lorentz transformations.

Finally, let's look at dilatations. The generator for dilatations is given here (2.33), whilst the current is given here (2.53). Put in the Ward identity, we get

$$\partial_\mu \langle T^\mu_\nu x^\nu X \rangle = -\sum_{k=i}^{n} \delta(x - x_k) \left[x^\nu_k \frac{\partial}{\partial x^\nu_k} \langle X \rangle + \Delta_k \langle X \rangle \right] \tag{2.59}$$

Once again, we can simplify using Leibnitz rule

$$\langle T^\mu_\mu X \rangle = -\sum_{k=i}^{n} \delta(x - x_k) \Delta_k \langle X \rangle \tag{2.60}$$

This is the Ward identity associated with dilatations.

We would now like to study the 2 dimensional case. We thus start by rewriting the Ward identities according to the complex coordinates defined here (1.13). We recall that in these coordinates, the metric is given by (1.14). Moreover, in these coordinates, the antisymmetric tensor takes the form

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2}i \\ -\frac{1}{2}i & 0 \end{pmatrix} \tag{2.61}$$

We will use the identity

$$\delta(x) = \frac{1}{\pi} \partial_{\bar{z}} \frac{1}{z} = \frac{1}{\pi} \partial_z \frac{1}{\bar{z}} \tag{2.62}$$

We will not prove this indentity, which can be derived from contour integrals. A detailed justification is given here [FMS97, p. 119]. Using these, we rewrite the Ward identity derived above

$$\begin{aligned}
2\pi \partial_z \langle T_{\bar{z}z} X \rangle + 2\pi \partial_{\bar{z}} \langle T_{zz} X \rangle &= -\sum_{k=1}^{n} \partial_{\bar{z}} \frac{1}{z - w_k} \partial_{w_k} \langle X \rangle \\
2\pi \partial_z \langle T_{\bar{z}\bar{z}} X \rangle + 2\pi \partial_{\bar{z}} \langle T_{z\bar{z}} X \rangle &= -\sum_{k=1}^{n} \partial_z \frac{1}{\bar{z} - \bar{w}_k} \partial_{\bar{w}_k} \langle X \rangle \\
2 \langle T_{z\bar{z}} X \rangle + 2 \langle T_{\bar{z}z} X \rangle &= -\sum_{k=1}^{n} \delta(x - x_k) \Delta_k \langle X \rangle \\
-2 \langle T_{z\bar{z}} X \rangle + 2 \langle T_{\bar{z}z} X \rangle &= -\sum_{k=1}^{n} \delta(x - x_k) s_k \langle X \rangle
\end{aligned} \tag{2.63}$$

Adding and substracting the 2 last equations of (2.63) and using (2.62), we have

$$\begin{aligned}
2\pi \langle T_{\bar{z}z} X \rangle &= -\sum_{k=1}^{n} \partial_{\bar{z}} \frac{1}{z - w_k} h_k \langle X \rangle \\
2\pi \langle T_{z\bar{z}} X \rangle &= -\sum_{k=1}^{n} \partial_z \frac{1}{\bar{z} - \bar{w}_k} \bar{h}_k \langle X \rangle
\end{aligned} \tag{2.64}$$

where $h_k = \Delta_k + s_k$ and $\bar{h}_k = \Delta_k - s_k$. Notice how in 2 dimensions, we naturally got back the possibility of 2 different scaling dimensions, due to the spin, as we saw in (1.25).

We renormalize the energy-momentum tensor as follows

$$T = -2\pi T_{zz} \qquad \bar{T} = -2\pi T_{\bar{z}\bar{z}} \tag{2.65}$$

Introducing the equation aboves in the first 2 equations of (2.63), we have

$$\begin{aligned} \langle T(z)X\rangle &= \sum_{i=1}^{n}\left(\frac{1}{z-w_k}\partial_{w_k}\langle X\rangle + \frac{h_i}{(z-w_k)^2}\langle X\rangle\right) + \text{reg.} \\ \langle \bar{T}(z)X\rangle &= \sum_{i=1}^{n}\left(\frac{1}{\bar{z}-\bar{w}_k}\partial_{\bar{w}_k}\langle X\rangle + \frac{\bar{h}_i}{(\bar{z}-\bar{w}_k)^2}\langle X\rangle\right) + \text{reg.} \end{aligned} \tag{2.66}$$

where reg. stands for an holomorphic function of z or $\bar{z}$, regular when $z \to w_k$ or $\bar{z} \to \bar{w}_k$.

Finally, we would like to bring the Ward identities associated to translations, Lorentz transformations and dilatations into a single Ward identity. We will not go into the details of the computation, but only go through the main idea of the proof. More details of this proof can be found here [FMS97, p. 121]. Let $x^\nu \to x^\nu + \epsilon^\nu(x)$ be an arbitrary conformal coordinate variation. We can write

$$\begin{aligned} \partial_\mu(\epsilon_\nu T^{\mu\nu}) &= \epsilon_\nu \partial_\mu T^{\mu\nu} + \frac{1}{2}(\partial_\nu\epsilon_\mu + \partial_\mu\epsilon_\nu)T^{\mu\nu} + \frac{1}{2}(\partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu)T^{\mu\nu} \\ &= \epsilon_\nu \partial_\mu T^{\mu\nu} + \frac{1}{2}(\partial_\rho\epsilon^\rho)\eta_{\mu\nu}T^{\mu\nu} + \frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha\epsilon_\beta\varepsilon_{\mu\nu}T^{\mu\nu} \end{aligned} \tag{2.67}$$

with $\varepsilon_{\mu\nu}$ the antisymmetric tensor. Using the 3 Ward identities on both sides, we have

$$\delta_\epsilon\langle X\rangle = \int \mathrm{d}^n\ w\mathrm{d}^2\ x\partial_\mu\langle T^{\mu\nu}(x)\epsilon_\nu(x)X\rangle \tag{2.68}$$

where the integral goes through all possible positions for the fields in the sequence X. Applying Gauss's theorem and using the notations introduced by (2.65), we finally get what is known as the *conformal Ward identity*:

$$\delta_{\epsilon,\bar{\epsilon}}\langle X\rangle = -\frac{1}{2\pi i}\oint_C \mathrm{d}z\ \epsilon(z)\langle T(z)X\rangle + \frac{1}{2\pi i}\oint_C \mathrm{d}\bar{z}\ \bar{\epsilon}(\bar{z})\langle \bar{T}(\bar{z})X\rangle \tag{2.69}$$

2.2.5 Operator product expansion

We notice in the Ward identities (2.66) that the correlation function diverges when $z \to w_k$ or $\bar{z} \to \bar{w}_k$. In general, it is typical for correlation functions to diverge when the position of two fields coincide. This reflects the infinite quantum fluctuations of a field taken at a precise position.

Due to the invariance of the theory through translations and Lorentz transformations, we have $\langle\phi_1(z)\phi_2(w)\rangle = f(|z-w|)$. f can be expanded in its Laurent serie, which in general will contain only a finite number of terms diverging when $x \to w$. We can represent each of these terms by an operators, well defined when $z \to w$, multiplied by a function diverging when $z \to w$. We call this representation the operator product expansion, or OPE. We usually write it by removing the brackets $\langle \dots \rangle$, without forgetting that it only makes sens inside of correlation functions.

For exemple, the Ward identities (2.66) can be rewritten for a primary field of conformal dimensions $h, \bar{h}$

$$\begin{aligned} T(z)\phi(w,\bar{w}) &\sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}) \\ \bar{T}(\bar{z})\phi(w,\bar{w}) &\sim \frac{h}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w,\bar{w}) \end{aligned} \tag{2.70}$$

2.3 Correlations in the free boson

Let's go back to the free boson theory, introduced in 1.3. As mentionned at the end of 1.3, we will now be considering the massless case. Using our new-found knowledge, we would like to compute the propagator of the free boson field, that is its 2-points correlation function.

$$K(x,y) = \langle \varphi(x)\varphi(y) \rangle \tag{2.71}$$

We can rewrite (1.33) as

$$S = \frac{1}{2}\int d^2x d^2y \varphi(x) A(x,y) \varphi(y) \qquad A(x,y) = -\partial^2\delta(x,y) \tag{2.72}$$

But (2.11) gives an expression of the correlation function according to the action. In fact, this expression can be reduced to $K(x,y) = A^{-1}(x,y)$, or equivalently

$$-\partial_x^2 K(x,y) = \delta(x,y) \tag{2.73}$$

This is a consequence of the properties of Gaussian integrals, which we will not go through but are discussed in [FMS97, p. 51].

Because of translation and rotation invariance, $K(x,y)$ should only depend on $|x-y|$. We thus write $K(x,y) = K(r)$. Integrating (2.73) over a disk centered around y then results in

$$1 = -2\pi r K'(r) \tag{2.74}$$

whose solution is

$$K(r) = -\frac{1}{2\pi}\ln(r) + C \tag{2.75}$$

with C a constant. This, in the usual coordinates, is equivalent to

$$\langle \varphi(x)\varphi(y) \rangle = -\frac{1}{4\pi}\ln(x-y)^2 + C \tag{2.76}$$

In the complex coordinates described in (1.13), this is

$$\langle \varphi(z,\bar{z})\varphi(w,\bar{w}) \rangle = \frac{1}{4\pi}(\ln(z-w) + \ln(\bar{z}-\bar{w})) + C \tag{2.77}$$

Taking the derivatives $\partial_z\varphi$ and $\partial_{\bar{z}}\varphi$, we can separate the holomorphic and antiholomorphic coordinates

$$\begin{aligned} \langle \partial_z\varphi(z,\bar{z})\partial_w\varphi(w,\bar{w}) \rangle &= \frac{1}{4\pi}\frac{1}{(z-w)^2} \\ \langle \partial_{\bar{z}}\varphi(z,\bar{z})\partial_{\bar{w}}\varphi(w,\bar{w}) \rangle &= \frac{1}{4\pi}\frac{1}{(\bar{z}-\bar{w})^2} \end{aligned} \tag{2.78}$$

For the remaining of this subsection, we will concentrate on the holomorphic dimension. Every equation we will derive in the holomorphic dimension will have a counterpart in the antiholomorphic dimension. We will write $\partial\varphi \equiv \partial_z\varphi$. We have seen just above that the OPE of $\partial\varphi$ with itself is

$$\partial\varphi(z)\partial\varphi(w) \sim -\frac{1}{4\pi(z-w)^2} \tag{2.79}$$

The fact that the two variables z and w are interchangeable show the bosonic nature of the field.

The energy-momentum tensor associated to the system is

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi \tag{2.80}$$

In complex coordinates, this is

$$T(z) = -2\pi : \partial\varphi(z)\partial\varphi(z) : \tag{2.81}$$

The normal-ordering here is necessary, to ensure the vanishing of the vacuum expectation value.

We can calculate the OPE of $T(z)$ with $\partial\varphi$ using Wick's theorem, knowing the OPE of $\partial\varphi$ with itself:

$$\begin{aligned} T(z)\partial\varphi(w) &= -2\pi : \partial\varphi(z)\partial\varphi(z) : \partial\varphi(w) \\ &\sim -2\pi : \partial\varphi(z)\partial\varphi(z) : \partial\varphi(w) - 2\pi : \partial\varphi(z)\partial\varphi(z) : \partial\varphi(w) : \partial\varphi(w) \\ &\sim \frac{\partial\varphi(z)}{(z-w)^2} \\ &\sim \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial_w^2\varphi(w)}{z-w} \end{aligned} \tag{2.82}$$

Recalling (2.70), this shows that $\partial\varphi$ is a primary field of conformal dimension $h = 1$.

Following the same methods, we can also compute the OPE of the energy-momentum tensor with itself:

$$T(z)T(w) \sim \frac{1}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{2.83}$$

This OPE does not respect (2.70), showing that the energy-momentum tensor is not a primary field.

Remark. In general, for any theory considered, we have

$$T(z)T(w) \sim \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{2.84}$$

We call c the central charge, which is equal to 1 for the free boson.

Chapter 3

Radial quantization

The radial quantization is a very useful quantization in conformal field theories. Moreover, it serves as a basis for computing later on orbifolds. In 3.1, we define what the radial quantization is. We then use it to give a more general form for fields in 3.2, which links back to 1.2.2. Finally, in 3.3, we quantize radially the free boson and show that this quantization allows for new primary fields.

3.1 Radial quantization

We stay in the context of 2-dimensional conformal field theories. Let's consider the space of our theory. It is a 2 dimensional Euclidian space, where one dimension corresponds to time whilst the other corresponds to space. We have canonical coordinates on this space, along two orthogonal axis, usually denoted by (x, y). One of the coordinates usually corresponds to the time dimension, whilst the other usually corresponds to the space dimension. Using these, we have defined our complex coordinates with (1.13), which have proven useful to distinguish the 2 chiral parts of our theory. In these coordinates, we can express time as $\frac{1}{2}(z + \bar{z})$, and space as $\frac{i}{2}(\bar{z} - z)$. However, the space and time are arbitrarily chosen. One could in fact choose any basis of the 2 dimensional space, and attribute time to one axis and space to the other axis. The only condition is that the basis must be orthogonal.

Moreover, as we are considering a conformal field theory, one can map 0 to ∞ and ∞ to 0 at will using symmetries. It would therefore be coherent to consider a compactified space, where we have a point for the two infinities in each dimension. Compactifying the space dimension, one could consider a cylinder $\mathbb{S}^1 \times \mathbb{R}$ of circumference L where $\mathbb{S}^1$ corresponds to the space dimension, and $\mathbb{R}$ corresponds to the time dimension. In this space, we have the canonical coordinates (x, t) where we identified (x, t) and $(x + L, t)$ for any x, t. Furthermore, adding a point at $-\infty$ in the time dimension, we can map this cylinder back to the complex plane through the map

$$(x, t) \to e^{\frac{2\pi}{L}(t+ix)} \tag{3.1}$$

This way, we have defined a new basis for our Euclidian space, and new space and time dimensions.

We want to define an Hermitian conjugation on this space, taking into account the newly defined time and space dimensions. To do so, let's consider an interacting field ϕ. Just like we saw in 2.1.1, we would like to compute the interactions happening in ϕ by considering a state ϕ_{in} "entering" the space at a time $-\infty$, and by computing its probability $\mathbb{P} = \langle \phi_{\text{out}} | \phi_{\text{in}} \rangle$ to become a state ϕ_{out} "exiting" the system at a time $+\infty$.

In the radial quantization, we can easily compute ϕ_{in}. We want

$$\phi_{\text{in}} \propto \lim_{t \to -\infty} \phi(x,t) \tag{3.2}$$

But in the complex plane, 0 corresponds to $t = -\infty$. We have

$$|\phi_{\text{in}}\rangle = \lim_{z,\bar{z} \to 0} \phi(z,\bar{z})|0\rangle \tag{3.3}$$

And

$$\langle \phi_{\text{out}}| = |\phi_{\text{in}}\rangle^\dagger \tag{3.4}$$

In the Euclidian formalism we have been using up until now, we considered a complex time $\tau = it$, such that under Hermitian conjugation time would transform as $\tau \to -\tau$, whilst space would be left unchanged. We would like to keep this kind of transformation. With radial quantization, reversing time while leaving space unchanged corresponds to the map $z \to \frac{1}{z*}$. One could be tempted to define Hermitian conjugation as follows

$$\phi(z,\bar{z})^\dagger = \phi(\frac{1}{\bar{z}}, \frac{1}{z}) \tag{3.5}$$

But if we did so, we would have

$$\begin{aligned} \langle \phi_{\text{out}} | \phi_{\text{in}} \rangle &= \lim_{z,\bar{z},w,\bar{w} \to 0} \langle 0 | \phi(z,\bar{z})^\dagger \phi(w,\bar{w}) | 0 \rangle \\ &= \lim_{z,\bar{z},w,\bar{w} \to 0} \langle 0 | \phi(\frac{1}{\bar{z}}, \frac{1}{z}) \phi(w,\bar{w}) | 0 \rangle \\ &= \lim_{z,\bar{z} \to +\infty} \langle 0 | \phi(\bar{z}, z) \phi(0,0) | 0 \rangle \end{aligned} \tag{3.6}$$

Supposing that ϕ is a primary field, recalling (2.19), we have

$$\langle 0 | \phi(\bar{z}, z) \phi(0,0) | 0 \rangle \propto z^{-2\bar{h}} \bar{z}^{-2h} \tag{3.7}$$

$\langle \phi_{\text{out}} | \phi_{\text{in}} \rangle$ would thus always be equal to 0, which is not what we want. In order to make sense of $\langle \phi_{\text{out}} | \phi_{\text{in}} \rangle$, we must define the Hermitian conjugate of a field as

$$\phi(z,\bar{z})^\dagger = \bar{z}^{-2h} z^{-2\bar{h}} \phi(\frac{1}{\bar{z}}, \frac{1}{z}) \tag{3.8}$$

3.2 Operator mode expansions

We want to expand a conformal field $\phi(z,\bar{z})$ with dimension $(h,\bar{h})$ in terms of a family of fields independent of the position. As the field is holomorphic in z and in $\bar{z}$, it may be written

$$\begin{aligned} &\phi(z,\bar{z}) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} z^{-m-h} \bar{z}^{-n-\bar{h}} \phi_{m,n} \\ &\text{with} \qquad \phi_{m,n} = \frac{1}{2\pi i} \oint \mathrm{d}z \; z^{m+h-1} \frac{1}{2\pi i} \oint \mathrm{d}\bar{z} \; \bar{z}^{n+\bar{h}-1} \phi(z,\bar{z}) \end{aligned} \tag{3.9}$$

Comparing the straightforward Hermitian conjugation of this expression with the Hermitian conjugation defined by (3.8), we get

$$\phi^\dagger_{m,n} = \phi_{-m,-n} \tag{3.10}$$

This is why we added the powers of h and $\bar{h}$ in (3.9) to define $\phi_{m,n}$.

Once again, we will now drop the antiholomorphic coordinate to simplify the notations. We must though not forget that it is always here, and can easily be restored. We have

$$\begin{aligned} \phi(z) &= \sum_{m\in\mathbb{Z}} z^{-m-h}\phi_m \\ \phi_m &= \frac{1}{2\pi i}\oint \mathrm{d}z\ z^{m+h-1}\phi(z) \end{aligned} \tag{3.11}$$

Let's now relate contour integrals to commutators in operator product expansions. Let $a(z)$ and $b(z)$ be two holomorphic fields. We consider the integral

$$\oint_w \mathrm{d}z\ a(z)b(w) \tag{3.12}$$

where the integral goes around w without getting around 0. This expression as a meaning inside of correlation functions, as long as it is time ordered. We can write

$$\oint_w \mathrm{d}z\ a(z)b(w) = \oint_{C_1} \mathrm{d}z\ a(z)b(w) - \oint_{C_2} \mathrm{d}z\ b(w)a(z) \tag{3.13}$$

where the two integrals on the right hand side are integrals at fixed time, meaning C_1 and C_2 are 2 circles centered around 0, and where C_1 wraps around w and C_2 doesn't.

Remark. Inside a correlation function, there are usually more fields than these two. This relation is true if the only field having a singularity inside of C_1 and not C_2 is $b(w)$. We should thus choose C_1 and C_2 as close as possible.

Defining the operator

$$A \equiv \oint a(z)\mathrm{d}z \tag{3.14}$$

we then have

$$\oint_w \mathrm{d}z\ a(z)b(w) = [A, b(w)] \tag{3.15}$$

Defining the operator B similarly to A, we can then integrate to obtain

$$[A, B] = \oint_0 \mathrm{d}w \oint_w \mathrm{d}z\ a(z)b(w) \tag{3.16}$$

Let's see how this relation can improve our understanding of the energy-momentum tensor.

As was done to get the conformal Ward identity (2.69), we let $\epsilon(z)$ be the holomorphic component of an infinitesimal conformal transformation. Defining the conformal charge

$$Q_\epsilon \equiv \frac{1}{2\pi i}\oint \mathrm{d}z\epsilon(z)T(z) \tag{3.17}$$

we can rewrite the conformal Ward identity using (3.15) as follows

$$\delta_\epsilon \phi(w) = -[Q_\epsilon, \phi(w)] \tag{3.18}$$

We see here that the conformal charge is a generator for conformal transformations, in a similar way as the transformation generators were defined by (2.23).

Using the mode expansion defined by (3.11), we can write

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \tag{3.19}$$

Also writing

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} \epsilon_n \tag{3.20}$$

we can rewrite (3.17) as

$$Q_\epsilon = \sum_{n\in\mathbb{Z}} \epsilon_n L_n \tag{3.21}$$

Remark. We have the exact same equations in the antiholomorphic dimension, defining the operators $(\bar{L}_n)_n$.

We can therefore see that the mode operators $(L_n)_n$ and $(\bar{L}_n)_n$ generates the conformal transformations on the Hilbert space. In particular, we see that L_0 and $\bar{L}_0$ generates the dilatations. But in radial quantization, dilatations correspond to time translations. As such, we get that the Hamiltonian H of our system verifies

$$H \propto L_0 + \bar{L}_0 \tag{3.22}$$

Using the relation between commutators in OPEs and contour integrals (3.16), the definition of the families $(L_n)_n$ and $(\bar{L}_n)_n$ (3.19), and the OPE of the energy-momentum tensor with itself (2.84), we obtain the commutation relation of the $(L_n)_n$ and $(\bar{L}_n)_n$:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m=-n} \\
[\bar{L}_m, L_n] &= 0 \\
[\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m=-n}
\end{aligned} \tag{3.23}$$

which exactly corresponds to 2 representations of the Virasoro algebra defined by (1.24). We have seen in the subsection 1.2.2 that any 2-dimensional conformal field theory contains 2 copies of the Virasoro algebra, one in each conformal dimension. We know see that these two algebras are indeed represented in any conformal field theory, and live in the energy-momentum tensor.

3.3 Quantization of the free boson

3.3.1 The free boson on a cylinder

Let's consider once again the theory of the free boson. We want to define the free boson field on a cylinder of circumference L, to eventually switch to the radial quantization. The theory stays the same, except the fact that we add a periodicity condition on the field

$$\varphi(x+L,t) = \varphi(x,t) \tag{3.24}$$

As the field is periodic, we can expand it using the Fourier transform

$$\begin{aligned} \varphi(x,t) &= \sum_n e^{2\pi i n x/L}\varphi_n(t) \\ \text{with} \quad \varphi_n(t) &= \frac{1}{L}\int \mathrm{d}x\; e^{-2\pi i n x/L}\varphi(x,t) \end{aligned} \tag{3.25}$$

Introducing the Fourier modes into the Lagrangian (1.33), we get

$$\mathcal{L} = \frac{1}{2}L\sum_n\left(\dot{\varphi}_n\dot{\varphi}_{-n} - \left(\frac{2\pi n}{L}\right)^2\varphi_n\varphi_{-n}\right) \tag{3.26}$$

Now, the momentum associated to ϕ_n is

$$\pi_n = L\dot{\varphi}_{-n} \tag{3.27}$$

We do still have $[\varphi_n,\pi_m] = i\delta_{n,m}$. We can write the Hamiltonian

$$H = \frac{1}{2L}\sum_n(\pi_n\pi_{-n} + (2\pi n)^2\varphi_n\varphi_{-n}) \tag{3.28}$$

From this, we note that

$$\varphi_n^\dagger = \varphi_{-n} \qquad \pi_n^\dagger = \pi_{-n} \tag{3.29}$$

The Hamiltonian corresponds once again to a sum of decoupled harmonic oscillators, of frequencies $\omega_n = \frac{2\pi|n|}{L}$. We see that the term $n=0$ vanishes. This is due to the absence of mass of the system, and is the cause of its conformal invariance. With a mass, the system wouldn't be invariant through conformal transformations.

Let's now define the creation $\tilde{a}_n^\dagger$ and annihilation $\tilde{a}_n$ operators. Usually, one would define them as follows

$$\tilde{a}_n = \frac{1}{\sqrt{4\pi|n|}}(2\pi|n|\varphi_n + i\pi_{-n}) \tag{3.30}$$

so that we can have the usual $[\tilde{a}_n,\tilde{a}_m] = 0, [\tilde{a}_n\tilde{a}_m^\dagger] = \delta_{n,m}$. However, this definition does not work with the zero mode.

Instead, we will treat the zero mode separately, and define the following operators

$$\begin{aligned} &\text{for } n>0, \quad a_n = -i\sqrt{n}\tilde{a}_n \quad \text{and} \quad \bar{a}_n = -i\sqrt{n}\tilde{a}_{-n} \\ &\text{for } n<0, \quad a_n = -i\sqrt{n}\tilde{a}_{-n}^\dagger \quad \text{and} \quad \bar{a}_n = -i\sqrt{n}\tilde{a}_n^\dagger \end{aligned} \tag{3.31}$$

These operators have the following commutation relations

$$[a_n,a_m] = n\delta_{n,m} \qquad [a_n,\bar{a}_m] = 0 \qquad [\bar{a}_n,\bar{a}_m] = n\delta_{n,m} \tag{3.32}$$

Writing the Hamiltonian in term of these operators, we have

$$H = \frac{1}{2L}\pi_0^2 + \frac{2\pi}{L}\sum_{n\geq 0}(a_{-n}a_n + \bar{a}_{-n}\bar{a}_n) \tag{3.33}$$

(3.32) leads to

$$[H, a_{-n}] = \frac{2\pi}{L}na_{-n} \tag{3.34}$$

which means that for $|E\rangle$ an eigenstate of H of energy E, $a_{-n}|E\rangle$ is still an eigenstate of H with energy $E + \frac{2n\pi}{L}$

Expressing the Fourier modes according to these operators, we have

$$\varphi_n = \frac{i}{n\sqrt{4\pi}}(a_n - \bar{a}_{-n}) \tag{3.35}$$

We can thus write at $t = 0$

$$\varphi(x) = \varphi_0 + \frac{i}{\sqrt{4\pi}}\sum_{n\neq 0}\frac{1}{n}(a_n - \bar{a}_{-n})e^{2\pi inx/L} \tag{3.36}$$

Thanks to the explicit expression of the Hamiltonian (3.33), we have

$$\varphi(x,t) = \varphi_0 + \frac{1}{L}\pi_0 t + \frac{i}{\sqrt{4\pi}}\sum_{n\neq 0}\frac{1}{n}\left(a_n e^{2\pi in(x-t)/L} - \bar{a}_{-n}e^{2\pi in(x+t)/L}\right) \tag{3.37}$$

Going to the Euclidian spacetime by replacing t with $-i\tau$ and using the complex coordinates

$$z = e^{2\pi(\tau - ix)/L} \qquad \bar{z} = e^{2\pi(\tau+ix)/L} \tag{3.38}$$

we finally have

$$\varphi(z,\bar{z}) = \phi_0 - \frac{i}{4\pi}\pi_0\ln(z\bar{z}) + \frac{i}{\sqrt{4\pi}}\sum_{n\neq 0}\frac{1}{n}(a_n z^{-n} + \bar{a}_n\bar{z}^{-n}) \tag{3.39}$$

Even though φ is not a primary field, we know from (2.82) that its derivatives $\partial\varphi$ and $\bar{partialvarphi}$ are. Let's concentrate on the holomorphic derivative.

Derivating the expansion of φ found just above, we have

$$i\partial\varphi(z) = \frac{\pi_0}{4\pi z} + \frac{1}{\sqrt{4\pi}}\sum_{n\neq 0}a_n z^{-n-1} \tag{3.40}$$

Introducing

$$a_0 \equiv \bar{a}_0 \equiv \frac{\pi_0}{\sqrt{4\pi}} \tag{3.41}$$

we can include the zero term in our sum, and write

$$i\partial\varphi(z) = \frac{1}{\sqrt{4\pi}}\sum_n a_n z^{-n-1} \tag{3.42}$$

Now that we have an explicit expression of $\partial\varphi$ we can also explicitely express $T(z)$. Recalling (2.81), we have

$$T(z) = \frac{1}{2}\sum_{n,m\in\mathbb{Z}} z^{-n-m-2} :a_n a_m: \tag{3.43}$$

which implies

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} a_{n-m}a_m \qquad \text{for } n \neq 0$$
$$L_0 = \sum_{n\geq 0} a_{-n}a_n + \frac{1}{2}a_0^2 \tag{3.44}$$

We can then rewrite (3.33) as

$$H = \frac{2\pi}{L}(L_0 + \bar{L}_0) \tag{3.45}$$

3.3.2 Vertex operators

Since $\partial\varphi$ has a scaling dimension of 1, φ has a vanishing scaling dimension. Making use of this, we can define a family of fields $(\mathcal{V}_\alpha)_{\alpha\in\mathbb{R}}$ without introducing any notion of scale, which we call the vertex operators:

$$\begin{aligned}\mathcal{V}_\alpha(z,\bar{z}) &\equiv : e^{i\alpha\phi(z,\bar{z})} : \\ &= e^{i\alpha\varphi_0 + \frac{\alpha}{\sqrt{4\pi}}\sum_{n\geq 0}\frac{1}{n}(a_{-n}z^n + \bar{a}_{-n}\bar{z}^n)} e^{\frac{\alpha}{4\pi}\pi_0 \ln(z\bar{z}) - \frac{\alpha}{\sqrt{4\pi}}\sum_{n\geq 0}\frac{1}{n}(a_n z^{-n} + \bar{a}_n\bar{z}^{-n})}\end{aligned} \tag{3.46}$$

We want to see how these newly defined fields act. Let's first calculate the OPE of $\partial\varphi$ with $\mathcal{V}_\alpha$, using the definition of the exponential and Wick's theorem, as was done to compute (2.82):

$$\begin{aligned}\partial\varphi(z)\mathcal{V}_\alpha(w,\bar{w}) &= \sum_{n=0}^{+\infty}\frac{(i\alpha)^n}{n!}\partial\varphi(z) : \varphi(w,\bar{w})^n : \\ &\sim \frac{-1}{4\pi(z-w)}\sum_{n=1}^{+\infty}\frac{(i\alpha)^n}{(n-1)!} : \varphi(w,\bar{w})^{n-1} : \\ &\sim \frac{i\alpha}{4\pi}\frac{\mathcal{V}_\alpha(w,\bar{w})}{z-w}\end{aligned} \tag{3.47}$$

Similarly, we compute its OPE with the energy-momentum tensor, using its explicit expression given by (2.81).

$$\begin{aligned}T(z)\mathcal{V}_\alpha(w,\bar{w}) &= -2\pi\sum_{n=0}^{+\infty}\frac{(i\alpha)^n}{n!} : \partial\varphi(z)\partial\varphi(z) :: \varphi(w,\bar{w})^n : \\ &\sim \frac{1}{8\pi(z-w)^2}\sum_{n=2}^{+\infty}\frac{(i\alpha)^n}{(n-2)!} : \varphi(w,\bar{w})^{n-2} : \\ &\quad + \frac{1}{z-w}\sum_{n=1}^{+\infty}\frac{(i\alpha)^n}{n!} n : \partial\varphi(z)\varphi(w,\bar{w})^{n-1} : \\ &\sim \frac{\alpha^2}{8\pi}\frac{\mathcal{V}_\alpha(w,\bar{w})}{(z-w)^2} + \frac{\partial_w\mathcal{V}_\alpha(w,\bar{w})}{z-w}\end{aligned} \tag{3.48}$$

The 2 terms here comes from the single and double contractions, from Wick's theorem. Due to this OPE, we see that the fields $\mathcal{V}_\alpha$ are primary fields, of holomorphic dimension

$$h(\alpha) = \frac{\alpha^2}{8\pi} \tag{3.49}$$

Knowing that $\mathcal{V}_\alpha$ has the same OPE with the antiholomorphic energy-momentum tensor, we also get

$$\bar{h}(\alpha) = h(\alpha) = \frac{\alpha^2}{8\pi} \tag{3.50}$$

Chapter 4

Orbifold generalities

This introduction aims at giving an overview of orbifolds. In 4.1, we mathematically define orbifolds. In 4.2, we then see how orbifolds generates a method for constructing new orbifolds from old ones using symmetries. Finally, in 4.3, we study conformal field theories on the torus, which is important in the context of orbifolds.

4.1 Mathematical construction

This sections aims at introducing the idea behind orbifolds to the reader. A more detailed discussion of the mathematical properties of orbifolds can be found here [au222].

4.1.1 Quotient space and manifolds

An orbifold is a construction coming from topology aiming at generalizing manifolds through quotient spaces. To fully understand the idea behind orbifolds, let's first recall some basics of topology.

Definition 4.1.1. An *Hausdorff* space $(X, \mathcal{T})$ is a space X equipped with a topology $\mathcal{T}$ where, for any $x, y \in X$, there exist $U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$

Definition 4.1.2. A *paracompact* space X is an Hausdorff space such that any cover $(U_k)_{k\in I}$ of X made by open subsets can be refined down to a locally finite family $(U_k)_{k\in I'\subset I}$ still covering X, meaning that $\bigcup_{k\in I'} U_k = X$ and that for any point $x \in X$, there exist an open subset V such that $x \in V$ and such that $\{U_k \ / \ k \in I', U_k \cap V \neq \emptyset\}$ is finite.

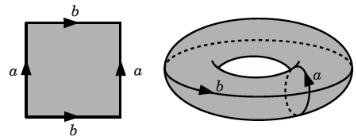
In the following, we will only consider Hausdorff spaces. The definition of paracompact spaces will serve later on. Now consider a topological space where we want to glue points together. We want some points to become topologically equivalent to others. We thus define an equivalence relation between points, and quotient the topological space by this equivalence relation, to keep only equivalence classes. The product is what we call a quotient space.

Definition 4.1.3. Let X be a space with topology $\mathcal{T}$, and let $\sim$ be an equivalence relation on X. The *quotient space* $X/\sim$ is the space of equivalence classes of X, equipped with the topology $\{\{[x] \ / \ x \in U\} \ / \ U \in \mathcal{T}\}$, where $[x]$ is the equivalence class of x.

For exemple, imagine we want to construct a torus. We start with a square in 2 dimensions, for exemple $[0,1]^2$. We then define the equivalence relation $\sim$ such that any point in the interior of the square has only itself in its equivalence class, and such that any point on a side is equivalent to the point on the other side of the square. Formally, this equivalence relation can be defined by

$$\begin{aligned} (x,0) &\sim (x,1) \quad \forall x \in [0,1] \\ (0,x) &\sim (1,x) \quad \forall x \in [0,1] \end{aligned} \tag{4.1}$$

The quotient space $[0,1]^2/\sim$ is then a torus.



Let us now suppose that we have a group G acting on our space X. We recall that the orbit of $x \in X$ is $Gx \equiv \{gx \ / \ g \in G\}$. Orbits give a partition of X. We can define an equivalence relation from this partition, and we write X/G the resulting quotient space. In a general fashion, quotienting a space by a group is more easy to write and visualize than quotienting a space by an equivalence relation. Both methods are often equivalent, but quotienting by groups is often preferred.

Let us take again the exemple of the torus. This time, we take $X = \mathbb{R}^2$. We take $G = \mathbb{Z}^2$, and we define its action on $\mathbb{R}^2$ as follows:

$$For(m,n) \in \mathbb{Z}^2, (x,y) \in \mathbb{R}^2, (m,n)(x,y) = (x+m, y+n) \tag{4.2}$$

With this, X/G is once again the torus.

We would now like to study the "smoothness" of our topological space. The smoothness reference should be a well known smooth space, such as $\mathbb{R}^n$. In this direction, we can naturally define a manifold as a topological space locally homeomorphic to $\mathbb{R}^n$.

Definition 4.1.4. An n-dimensional *topological manifold* M is a topological Hausdorff space which can be covered by a countable number of open subsets each homeomorphic to $\mathbb{R}^n$. This means that we dispose of a sequence of open subsets $(U_\alpha)_{\alpha\in\mathbb{N}}$ and a sequence of homeomorphisms $(\varphi_\alpha)_{\alpha\in\mathbb{N}}$ where each φ_α maps U_α in $\mathbb{R}^n$.

Remark. We call each map φ_α a chart, U_α its domain, and we call the set of all charts an atlas.

Let M be a manifold, and $(U_\alpha)_{\alpha\in\mathbb{N}}$ its domains. Let α, β such that $U_\alpha \cap U_\beta \neq \emptyset$. We can define

$$\begin{aligned} \varphi_{\alpha\beta} &= \varphi_\beta \circ \varphi_\alpha^{-1}|_{U_\alpha\cap U_\beta} \\ \varphi_{\beta\alpha} &= \varphi_\alpha \circ \varphi_\beta^{-1}|_{U_\alpha\cap U_\beta} \end{aligned} \tag{4.3}$$

$\varphi_{\alpha\beta}$ and $\varphi_{\beta\alpha}$ are called transition maps. The transition maps give important information about the smoothness of the manifold. In particular, we call a *differentiable manifold* a manifold equipped with an atlas where all transition maps are differentiables. Within differentiable manifolds, we call a C^k*-manifold* a manifold equipped with an atlas where all transition maps are of class C^k. Finally, we call a *smooth manifold* a manifold equipped with an atlas where all transition maps are infinitely differentiable.

4.1.2 Orbifolds

We would now like to equip quotient spaces with a smoothness structure, as done just above. The natural way of doing so would be to quotient a manifold. However, in general, the quotient of a manifold by a group is not a manifold. We call the product an orbifold. As for manifolds, we will define orbifold charts and orbifold atlases. Moreover, an other data, embeddings, will be define to ensure the compatibility of the action of the group between the charts.

Definition 4.1.5. Let X a topological space and $n \in \mathbb{N}$. We call an *orbifold chart* $(\tilde{U}, H, \phi)$ of dimension n a connected open subset $\tilde{U}$ of $\mathbb{R}^n$, a finite groupe H acting effectively on $\tilde{U}$ (meaning that the only element of H acting as the identity on $\tilde{U}$ is the neutral element of H), and a map $\phi : \tilde{U} \to U \subset X$ such that for all $h \in H$, $x \in \tilde{U}$, $\phi(hx) = \phi(x)$, and such that ϕ induces an homeomorphism between $\tilde{U}/H$ and U.

An orbifold chart is the right way to generalize manifold charts, but between $\mathbb{R}^n/H$ and U.

Definition 4.1.6. An *embedding* $\lambda : (\tilde{U}_1, H_1, \phi_1) \to (\tilde{U}_2, H_2, \phi_2)$ is an infinitely differentiable injective map between $\tilde{U}_1$ and $\tilde{U}_2$, such that $\phi_2 \circ \lambda = \phi_1$.

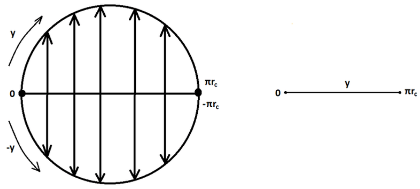
Definition 4.1.7. An *orbifold atlas* for X of dimension n is the data of a family $(\tilde{U}_k, H_k, \phi_k)_{k \in I}$ of orbifold charts of dimension n such that for any two charts $(\tilde{U}_1, H_1, \phi_1)$ and $(\tilde{U}_2, H_2, \phi_2)$, for $x \in U_1 \cap U_2$, there exist an open neighbourhood $U_3 \subset U_1 \cap U_2$ of x such that there is an orbifold chart $(\tilde{U}_3, H_3, \phi_3)$ in the family $(\tilde{U}_k, H_k, \phi_k)_{k \in I}$, and such that $(\tilde{U}_3, H_3, \phi_3)$ can be embedded in $(\tilde{U}_1, H_1, \phi_1)$ and $(\tilde{U}_2, H_2, \phi_2)$.

Definition 4.1.8. A *smooth orbifold* $\mathcal{O}$ is a Hausdorff paracompact topological space associated with an orbifold atlas.

We immediatly notice that any smooth manifold is an orbifold, where the $(H_k)_k$ are all trivial.

Let's see an exemple of orbifold. Let the topological space be $\mathbb{S}^1$, the circle. Now, let $\mathbb{Z}^2$ act on $\mathbb{S}^1$ such that for $(x, y) \in \mathbb{S}^1$, $1(x, y) = (x, -y)$. The orbifold resulting of this action is simply a segment.

We notice that a segment is not a manifold, as it has borders. We will come back to this exemple later.



4.2 Orbifolds and global symmetries

4.2.1 Orbifolds in conformal field theories

Now, let's go back to our main topic, which is conformal field theories. Let us consider a bosonic 2D conformal field theory $\mathcal{C}$, with a single vacuum, such that the theory admits a symmetry represented by the finite group G acting on the target space. This means that to the unitary representation of the conformal transformations group in the Hilbert space $\mathcal{H}$, we additionally have a unitary representation ρ of G in $\mathcal{H}$, such that for any $g \in G$, $\rho(g)$ fixes the vacuum, commutes with the Virasoro algebra, and preserves the operator product expansion. To avoid any confusion, we precise here that the considered symmetry (and the quotient) is on the the target space, the space on which the fields act, and not on the spacetime.

We recall the state/operator correspondence mentionned in 1.2.4. The Hilbert space can be freely viewed as the space of all possible states of the system, or as the space of all operators of the conformal field theory. We will therefore freely speak of operator product expansion in the Hilbert space. We call the orbifold construction the method of constructing a new conformal field theory $\mathcal{C}/G$, the orbifold of $\mathcal{C}$ by G, by quotienting the Hilbert space by its representation of G. Let us first suppose that G is abelian. We can describe the method in two steps:

- The first step is the *projection*, and consists in projecting all states on the subspace $\mathcal{H}^G \subset \mathcal{H}$ of states that are invariant under the action of G. From our hypothesis, we know that this subspace is not empty, as it contains at least the vacuum in its states, and the Virasoro algebra in its operators. Moreove, also by hypothesis, $\mathcal{H}^G$ is closed under operator product expansion.

- The second step is the construction of the *sector spaces*. For any sequence of fields $\varphi_1, \varphi_2, \dots \varphi_n$ in our theory, the correlation function $\langle \varphi_1 \dots \varphi_n \rangle$ may have branch cuts. More specifically, let us consider the function $f(z) = \langle \varphi_1(0)\varphi_2(z)\dots\rangle$. If we continue analytically $f(z)$ such that we can turn around 0 with z, $f(z)$ is not single valued as $z \to e^{2\pi i}z$. Indeed, we may have $g \in G$ such that $\varphi_2(e^{2\pi i}z) = \rho(g)\varphi_2(z)$. For $g \in G$, we construct the sector space $\mathcal{H}_g^G$ as the set of all fields φ_1 acting this way. We note that each twisted space sector contains a stable representation ρ_g of G projected from ρ.

Then, writing $\mathcal{H}_1^G = \mathcal{H}^G$, the space of all states of the new conformal field theory can be written as

$$\tilde{\mathcal{H}} = \oplus_{g \in G} \mathcal{H}_g^G \tag{4.4}$$

It is important to separate the sectors. Indeed, each one of them is closed under OPE, and can be studied separately. However, in the whole theory, due to the action of G on the states and operators, we will have operators allowing one to go from a sector to another.

We can generalize this construction to the case where G is not abelian. If G is not abelian, the projected action of ρ on the sector spaces is stable on every conjugacy class. We thus want to construct for all conjugacy class $[g]$ the associated sector space

$$\mathcal{H}_{[g]}^{G} \equiv \oplus_{h\in[g]}\mathcal{H}_{h}^{G} \tag{4.5}$$

on which there is a stable representation of G. The orbifold Hilbert space is then given by the sum

$$\tilde{\mathcal{H}} = \oplus_{[g]}\mathcal{H}_{[g]}^{G} \tag{4.6}$$

where $[g]$ goes through all conjugacy classes. However, this construction does not always yield a consistent CFT. Moreover, if a consistent orbifold construction exists for a given group G, it might not be unique. For the interested reader, a more detailed discussion on this matter can be found here [GV23, p. 9]

4.2.2 The space group

In the two following subsections, we will follow [Dix+87]. We place ourselves in a conformal field theory with an n-dimensional target space, meaning the fields act on an n-dimensional space. Let's consider a group S constituted of transformations (θ, a) acting as

$$x \to \theta x + a \tag{4.7}$$

where θ parameterize a rotation and a a translation. We will call S the space group. We are interested in the study of $\Omega = \mathbb{R}^n/S$.

We can write the multiplication and the inverse in S:

$$\begin{aligned}(\theta_1, a_1)(\theta_2, a_2) &= (\theta_1\theta_2, a_1 + \theta_1 a_2)\\ (\theta, a)^{-1} &= (\theta^{-1}, -\theta^{-1}a)\end{aligned} \tag{4.8}$$

The space group is the product of a group of pure rotations P and of a group of pure translations T. We identify T to the set of points the origin can go to through the action of T, in $\mathbb{R}^n$. As we want Ω to be non-trivial, T should not be dense in $\mathbb{R}^n$. As it is a discrete subgroup of $\mathbb{R}^n$, it should be equal to a lattice. We thus have

$$\Omega = \mathbb{R}^n/T/\bar{P} = \mathbb{T}^n/\bar{P} \tag{4.9}$$

with $\mathbb{T}^n$ the torus in n dimensions, and $\bar{P}$ the projection of P on the torus. As S is a group, $\bar{P}$ should be a subgroup of the isometry group of the torus. Moreover, for the same reason as for T, $\bar{P}$ should be discrete, and thus finite.

S is not an abelian group. According to (4.6), we should thus have as many sectors as conjugacy classes in S. In each sectors, fields obey a different boundary condition. We separate the sectors in 2 categories. We define the *winding* sectors as the sectors corresponding to the conjugacy class of translations. Indeed, the conjugate of a translation always is a translation. For any field φ in such a sector, we have $a \in T$ so that

$$\varphi(e^{2\pi i} z) = \varphi(z) + a \tag{4.10}$$

We also define the *twisted* sectors as the sectors corresponding to conjugacy classe of rotations. We note that in the conjugacy class of a pure rotation, there are more than pure rotations. In these sectors, for any field φ, we have $g \in S$ such that

$$\varphi(e^{2\pi i} z) = g\varphi(z) \tag{4.11}$$

But we also have for $h \in S$

$$h\varphi(e^{2\pi i} z) = hg\varphi(z) = (hgh^{-1})h\varphi(z) \tag{4.12}$$

$h\varphi$ is still in the same sector, as it is the product of the action of S on a field of the sector. Writing $\varphi' = h\varphi$, we see that in a sector with fields having for boundary condition $\varphi(e^{2\pi i} z) = g\varphi(z)$, we also have fields having for boundary conditions

$$\varphi'(e^{2\pi i} z) = (hgh^{-1})\varphi'(z) \tag{4.13}$$

This proves, once again, that in the case of a non-abelian group G we have to consider the sector associated to conjugacy classes. For each twisted sector, we call the *twist operator* the operator leading a field from $\mathcal{H}_1^S$ to the twisted sector space.

Remark. For each of these boundary conditions, we have supposed the space-time to be in the radial quantization, such that going from $\varphi(z)$ to $\varphi(e^{2\pi i} z)$ corresponds to going around the space dimension, to the same point. In a different formalism, the boundary condition will be different. For exemple, in the cylinder formalism, for a cylinder of circumference L, we need to replace $\varphi(e^{2\pi i} z)$ by $\varphi(z + L)$

To illustrate this, let's consider once again the exemple of the one dimensional $\mathbb{Z}_2$ orbifold. Let's try to construct it by acting with a well-chosen space group on $\mathbb{R}$. We first want to construct the circle used as the base space in the exemple of 4.1.2. We thus include translations by an integer to S:

$$\{(1, n) \ / \ n \in \mathbb{Z}\} \subset S \tag{4.14}$$

Moreover, we want to identify one side of the circle with the other. To do so, we can cut our circle at the point associated to the origin of the real line. From this, we associate one side of the circle to the other, or pulling back our action to the real line, we associate one side of the real line to the other. This is equivalent to having the rotation $(-1, 0)$ as part of S. A general element of S therefore looks like $(\pm 1, n)$ with $n \in \mathbb{Z}$. We have $T = \mathbb{Z}$ and $P = \mathbb{Z}_2$. This is why we call it the $\mathbb{Z}_2$ orbifold. conjugating an element of S by any other element of S, we compute the conjugacy classes of the space group:

$$\begin{aligned}(1,m)(1,n)(1,m)^{-1} &= (1,m+n)(1,-m)\\ &= (1,n)\\ (-1,m)(1,n)(-1,m)^{-1} &= (-1,m-n)(-1,m)\\ &= (1,-n)\\ (1,m)(-1,n)(1,m)^{-1} &= (-1,m+n)(1,-m)\\ &= (1,n+2m)\\ (-1,m)(-1,n)(-1,m)^{-1} &= (1,m-n)(-1,m)\\ &= (-1,2m-n)\end{aligned} \tag{4.15}$$

As such, the translation conjugacy classes of S are $[(1,n)] = \{(1,n),(1,-n)\}$. To each of these class is associated a winding sector. Moreover, there are only two conjugacy classes for rotations, the rotations with an even translation $\{(-1,n)\ /\ n = 2k\}$ and those with an odd translation $\{(-1,n)\ /\ n = 2k+1\}$. These two conjugacy classes lead to two twisted sectors, and therefore two twist operators leading to these sectors.

We see better what happens when looking at the $\mathbb{Z}_2$ orbifold from the perspective of the circle, given in 4.1.2. We take $\mathbb{S}^1$ and identify the two sides of the circle $x = -x$. The winding sectors are simply due to the nature of the circle. Moreover, this transformation leaves two fixed points, given in the complex coordinates by $x = i$ and $x = -i$. Each of these fixed points create a singularity, leading to the two twisted sectors.

4.2.3 Global monodromy conditions

We now restrain ourselves to 2 dimensions. We can easily generalize the $\mathbb{Z}_2$ orbifold to the $\mathbb{Z}_\mathbb{N}$ orbifold, wher the group of rotations of S is $\mathbb{Z}_N$. We know that $\mathbb{Z}_N = e^{\frac{2i\pi k}{N}}$ is generated by a single rotation, $\theta = e^{\frac{2i\pi}{N}}$. A general element of the space group thus takes the form (θ^j, a), with $j \in \mathbb{N}_N$. The conjugacy class of a translation $(1, a_0)$ is $\left\{(1, \theta^j a_0)\ /\ j \in \mathbb{N}_N\right\}$. The winding sectors are described by these conjugacy classes. For $j \in \mathbb{N}_N$, there are also several conjugacy classes of the form $\left\{(\theta^j, a)\ /\ a \in \Lambda\right\}$, parametrized by a coset Λ of the lattice T. To each of these conjugacy class is associated a twisted sector. Let's try to see what forms can take these cosets. To do so, we conjugate (θ^j, a_0), an arbitrary element of a conjugacy class.

$$\begin{aligned}(\theta^k, a)(\theta^j, a_0)(\theta^k, a)^{-1} &= (\theta^{k+j}, a + \theta^k a_0)(\theta^{-k}, -\theta^{-k} a)\\ &= (\theta^j, \theta^k a_0 + (1-\theta^j)a)\end{aligned} \tag{4.16}$$

Therefore, the coset Λ_{a_0} of the conjugacy class of (θ^j, a_0) is equal to

$$\left\{\theta^k a_0 + (1-\theta^j)a\ /\ k \in \mathbb{Z}, a \in T\right\} \tag{4.17}$$

We will now focus on the singly-twisted and singly-antitwisted sectors, $j \pm 1$. We want to study the fixed points of these sectors. Considering (θ, a_0), let p_1,

p_2 two elements fixed by the conjugacy class of (θ, a_0). For any k_1, k_2, a_1, a_2,we have

$$\begin{aligned} p_1 &= \theta p_1 + \theta^{k_1} a_0 + (1-\theta) a_1 \\ p_2 &= \theta p_2 + \theta^{k_2} a_0 + (1-\theta) a_2 \end{aligned} \tag{4.18}$$

So

$$(1-\theta)(p_1 - p_2) = (\theta^{k_1} - \theta^{k_2}) a_0 + (1-\theta)(a_1 - a_2) \tag{4.19}$$

Assuming θ doesn't have eigenvalues equal to 1, p_1 and p_2 therefore only differ by a vector of T. Recalling (4.9), we see that p_1 and p_2 are projected onto the same point on the torus . Each conjugacy class can be associated to a point on $\mathbb{T}^2$ fixed by the rotation θ. Saying it another way, the "sector twisted by θ" has a subsector for each fixed point of θ on $\mathbb{T}^2$. This result is also easily shown to be true with the rotation θ^{-1}.

Rewriting (4.18), we see that

$$a_0 = (1-\theta)\theta^{-k_1} p_1 - (1-\theta)\theta^{-k_1} a_1 \tag{4.20}$$

But p_1 is fixed by the rotation θ. We can thus write the coset associated to the fixed point p:

$$\Lambda_p^+ = (1-\theta)(p+T) \tag{4.21}$$

Similarly, in the "sector twisted by θ^{-1}", we can associate to each fixed point p of θ^{-1} a subsector, whose conjugacy class is associated to the following coset

$$\Lambda_p^- = (1-\theta^{-1})(p+T) \tag{4.22}$$

In general, the sectors associated with twists $\theta^j, j \neq \pm 1$ do not correspond exactly to the fixed points of θ^j. The reason is that to get (4.21), we had to use the fact that the fixed point considered was fixed by any θ^k. For $j \neq \pm 1$, a fixed point of θ^j is not necessarily fixed by all θ^k. There is thus a very large class of twisted sectors, and twist fields, in the orbifold $\mathbb{Z}_N$. We can however label them with 2 indexes (j, ϵ), j corresponding to the twist θ^j of the sector, the other corresponding to the coset associated to the conjugacy class of the sector. In particular, we may write $\sigma_{j,\epsilon}$ the twist operator and $\mathcal{H}^S_{j,\epsilon}$ the twisted sector space associated to the twisted sector (j, ϵ). For $j \pm 1$, ϵ denotes the associated fixed point of θ^j.

Now that we have classified the twisted sectors by their conjugacy class, we can write the boundary condition of fields in the twisted sector spaces. Let (j, ϵ) be a twisted sector, and let $\varphi \in \mathcal{H}^S_{j,\epsilon}$. In a correlation function with the twist operator $\sigma_{j,\epsilon}(0)$, when moving $\varphi(z, \bar{z})$ around 0, we get the following condition

$$\varphi(e^{2\pi i} z, e^{-2\pi i} \bar{z}) = e^{\frac{2i\pi j}{N}} \varphi(z, \bar{z}) + a \tag{4.23}$$

with $a \in \Lambda_\epsilon$ the coset associated with ϵ. We call this condition the global monodromy condition.

4.3 Modular invariance

We have seen up to now the importance of the torus. In particular, (4.9) shows that the construction of any conformal field theory via orbifold first needs to go through the definition of a conformal field theory with values on a torus, meaning having a torus as the target space. Quotienting a conformal field theory $\mathcal{C}$ by a space group S composed of translations T and rotations P amounts to quotienting the conformal field theory having values on the torus $\mathcal{C}/T$ by the group of rotations, P.

Moreover, as discussed in 3.1, we can change the considered spacetime to, for exemple, in the exemple seen in this section, a cylinder. In the case of orbifolds, we would like to consider a torus for the spacetime, which corresponds to the usual spacetime where we have compactified the space and time dimensions. It is natural to believe that choosing the same topology for the spacetime and the target space can be very helpful. For these reasons, we will now study conformal field theories defined on the torus.

4.3.1 The modular group

A torus $\mathbb{T}^2$ results of the quotient of $\mathbb{R}^2$ by a lattice Λ. We may thus define a torus by two linearly independent vectors v_1 and v_2 generating the lattice Λ. On the complex plane, these two vectors may be represented by two complex numbers, ω_1 and ω_2. We call these complex numbers the periods of the lattice. Consider a conformal field theory living on the torus defined by (ω_1, ω_2). The conformal field theory is invariant through dilatations and through rotations. As such, the relative scale of the lattive or its orientation doesn't affect the theory. the conformal field theory only depends on the ratio $\tau = \frac{\omega_2}{\omega_1}$, called the *modular ratio*.

However, τ does not uniquely constraints a conformal field theory on the torus. Two sets of periods may define the same lattice, but have different modular ratios.

Let (ω_1, ω_2) and (ω_1', ω_2') two sets of periods defining the same lattice. Then ω_1' and ω_2' should both be in the lattice generated by (ω_1', ω_2'), meaning they must be expressible as an integer combination of ω_1 and ω_2. Reciprocally, ω_1 and ω_2 should both be expressible as an integer combination of ω_1' and ω_2'. This amounts to saying that there exist $a, b, c, d \in \mathbb{Z}$ such that

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \tag{4.24}$$

And reciprocally so. The fact that it must also be true reciprocally means that the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ must be inversible. More precisely, for the two lattices (ω_1, ω_2) and (ω_1', ω_2') to be equivalent, their unit cell should have the same area. Therefore, the determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ should be unity. The group of such transformations is the group of integer 2 by 2 matrices with unit determinant, that is $SL(2, \mathbb{Z})$.

Under the change of period described by (4.24), the modular parameter transforms as

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{4.25}$$

We notice that we can change the sign of all a, b, c, d simultaneously and still get the same transformation. We can therefore restrain the considered group of transformations to $SL(2,\mathbb{Z})/\mathbb{Z}_2$, or equivalently $PL(2,\mathbb{Z})$. This group is called the *modular group*. In the future, we will compute quantities of our conformal field theories, such as the partition function of a theory, according to the modular ratio τ. Due to what we just saw, these quantities will need to be invariant under the modular group. In a more general way, a conformal field theory defined on a torus will need to be modular invariant according to τ. This modular invariance will impose strong constraints on the system, much as conformal invariance imposed a lot of constraints on conformal field theories. This explains the name of this section, "Modular invariance", and justifies the importance of modular invariance in orbifold conformal field theories.

For now, let's try to see what the modular group looks like. The modular group is generated by these two transformations

$$\begin{aligned} \mathcal{T}: \tau \to \tau + 1 \qquad &\text{or} \qquad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ \mathcal{S}: \tau \to -\frac{1}{\tau} \qquad &\text{or} \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned} \tag{4.26}$$

Satisfying the following equations

$$(\mathcal{ST})^3 = \mathcal{S}^2 = 1 \tag{4.27}$$

We can interpret these two generators as follows. We write

$$\mathcal{U} = \mathcal{TST} : \tau \to \frac{\tau}{\tau + 1} \tag{4.28}$$

Considering a torus defined by the periods ω_1 and ω_2, applying $\mathcal{T}$ amounts to transforming the periods as

$$(\omega_1, \omega_2) \to (\omega_1, \omega_1 + \omega_2) \tag{4.29}$$

whilst applying $\mathcal{U}$ amounts to transforming the periods as

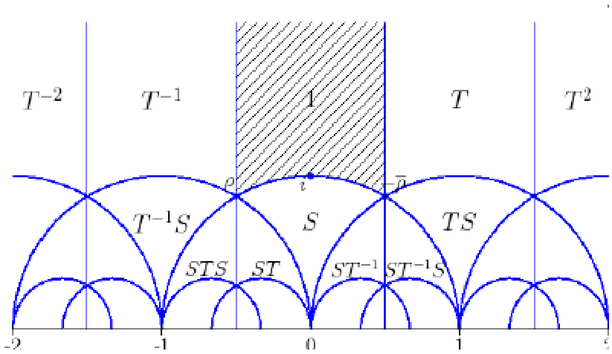
$$(\omega_1, \omega_2) \to (\omega_1 + \omega_2, \omega_2) \tag{4.30}$$

Each time, the transformation corresponds to cutting the torus at a fixed time or space, rotating one side by 2π, and gluing back the two ends together.

Now, let's see how modular transformations act on the complex plane. From (4.26), it is obvious that the modular group preserves the upper-half plane $\mathbb{H}$.

Definition 4.3.1. A *fundamental domain* Γ of a group G acting on a space is a connected subset of this space such that no pair of points in Γ can be connected through the action of G, but any point outside of Γ in the space can be reached by a unique point of the fundamental domain through the action of G.

Finding a fundamental domain of a group helps a lot in the general understanding of the group. In our case, the modular group acts on the upper-half plane $\mathbb{H}$. A common fundamental domain of the modular group is the subset of points with real part between $-\frac{1}{2}$ and $-\frac{1}{2}$, imaginary part higher than 0, and with a norm greater than 1. To make this a well defined fundamental domain, we take the points with norm strictly greater than 1 for points with a real part strictly greater than 0, and a norm greater or equal to 1 for points with an imaginary part less than 0. This fundamental domain is represented as the hatched part in the drawing below. In the drawing are also represented the first few actions of the generators to the fundamental domain.



4.3.2 Modular functions

Before diving into the calculations of a conformal field theory on a torus, we first digress a bit to introduce some functions that will become useful when manipulating modular invariant quantities. We will give some identities and how these functions transform under the modular group, without proving it. The proofs are not too difficult but are quite long. For the interested reader, they can be found here [FMS97, p. 390].

We first introduce the theta functions, which arise from solution of the heat equation or from the elliptic functions theory. For $\tau \in \mathbb{H}$, we write $q = e^{2\pi i\tau}$. We define

$$\begin{aligned}
\Theta_2(\tau) &= \sum_{n\in\mathbb{Z}} q^{(n+\frac{1}{2})^2/2} = 2q^{\frac{1}{8}} \prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2 \\
\Theta_3(\tau) &= \sum_{n\in\mathbb{Z}} q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}})^2 \\
\Theta_3(\tau) &= \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-\frac{1}{2}})^2
\end{aligned} \tag{4.31}$$

These functions transform as follows under the modular group

$$\begin{aligned}
\Theta_2(\tau+1) &= e^{\frac{\pi i}{4}}\Theta_2(\tau) \qquad & \Theta_2(-\frac{1}{\tau}) &= \sqrt{\frac{\tau}{i}}\Theta_4(\tau) \\
\Theta_3(\tau+1) &= \Theta_4(\tau) \qquad & \Theta_3(-\frac{1}{\tau}) &= \sqrt{\frac{\tau}{i}}\Theta_3(\tau) \\
\Theta_4(\tau+1) &= \Theta_3(\tau) \qquad & \Theta_4(-\frac{1}{\tau}) &= \sqrt{\frac{\tau}{i}}\Theta_2(\tau)
\end{aligned} \tag{4.32}$$

We now define Dedekin's η function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) \tag{4.33}$$

η transforms as follows under the modular group

$$\eta(\tau+1) = e^{\frac{\pi i}{12}}\eta(\tau) \qquad \eta(-\frac{1}{\tau}) = \sqrt{\frac{\tau}{i}}\eta(\tau) \tag{4.34}$$

Linking the theta functions and Dedekin's function, we have the following identity

$$\eta^3(\tau) = \frac{1}{2}\Theta_2(\tau)\Theta_3(\tau)\Theta_4(\tau) \tag{4.35}$$

4.3.3 Partition functions on the torus

Let's imagine a conformal field theory, living on the torus defined by the periods ω_1, ω_2. This does not necessarily mean that the conformal fields are periodic according to the periods. In the path integral formalism, it simply means that the action is invariant when translated by a period. Depending on the action, some freedom may be given to the field, resulting in different possible boundary conditions. For exemple, if the action of a field φ is quadratic in φ and $\dot{\varphi}$, φ could pick up a sign -1 when translated accross a period. In this case, there would be 4 possible boundary conditions

$$\begin{array}{rl} \varphi(z+\omega_1) = \varphi(z) & \varphi(z+\omega_2) = \varphi(z) \\ \varphi(z+\omega_1) = -\varphi(z) & \varphi(z+\omega_2) = \varphi(z) \\ \varphi(z+\omega_1) = \varphi(z) & \varphi(z+\omega_2) = -\varphi(z) \\ \varphi(z+\omega_1) = -\varphi(z) & \varphi(z+\omega_2) = -\varphi(z) \end{array} \tag{4.36}$$

leading to 4 different sectors. This is exactly the same phenomenon as was observed in the section 4.2.2, leading to the boundary condition (4.10). Indeed, the torus is defined by $\mathbb{R}^2/S$ where S is the space group generated by the translations by ω_1 and ω_2.

Given a boundary condition and supposing the boundary condition leaves the action invariant, we have a well defined conformal field theory on the torus, which can be formulated in the path-integral formalism. Knowing that, we would like to compute correlation functions. As in statistical mechanics, (2.11) shows us that the most important quantity to know in order to compute correlaton functions is the normalization constant of the correlation functions, the partition function Z. If we can compute the partition function Z of a conformal field theory on the torus, and if we know the correlation functions of the same theory on the infinite spacetime, we should be able to compute the correlation functions by renormalizing the correlation functions. This is especially useful when constructing a conformal field theory using orbifolds, where we already know the details of our conformal field theory befor quotienting it.

We continue the analogy with statistical mechanics. In statistical mechanics, one has

$$Z = \mathrm{Tr}(e^{-\frac{H}{kT}}) \tag{4.37}$$

with H the Hamiltonian. We would like to find a similar formula.

We define T_ω to be the operator translating the system by a period ω of the torus. Tracing back the computations from (2.10) to (2.8), we see that $Z = \mathrm{Tr}(T_\omega)$. We thus want to compute T_ω. We haven't yet fixed the time and space dimensions on the torus. As was discussed in the section on radial quantization, we can choose any orthogonal basis. We choose the space dimension to be along the real axis, and the time dimension to be along the imaginary axis. Let H be the Hamiltonian, generating translations in time, and P be the total momentum, generating translations in space. We have

$$T_\omega = e^{-H\mathrm{Im}(\omega)+i} \tag{4.38}$$

Let's now see our torus as a cylinder of length ω, with a circumference $L = \omega'$, ω' being the other period of the torus. We recall the relation (3.45). We write the same relation but shifting the Hamiltonian by a constant, to ensure the vanishing of the vacuum energy density when $L \to \infty$

$$H = \frac{2\pi}{L}(L_0 + \bar{L}_0 - \frac{c}{12}) \tag{4.39}$$

We can also find

$$P = \frac{2\pi i}{L}(L_0 - \bar{L}_0) \tag{4.40}$$

Injecting all of these into (4.37) and introducing the modular parameter $\tau = \frac{\omega}{\omega'}$, we get

$$Z(\tau) = \mathrm{Tr}(e^{2\pi i\left[\tau\left(L_0 - \frac{c}{24}\right) - \bar{\tau}\left(\bar{L}_0 - \frac{c}{24}\right)\right]}) \tag{4.41}$$

Introducing as in 4.3.2

$$q = e^{2\pi i\tau} \qquad \bar{q} = e^{2\pi i\bar{\tau}} \tag{4.42}$$

we finally write

$$Z(\tau) = \mathrm{Tr}(q^{L_0 - \frac{c}{24}}\bar{q}^{\bar{L}_0 - \frac{c}{24}}) \tag{4.43}$$

Chapter 5

Orbifolds of the free boson

In this section, we apply all precedent results to construct orbifolds of the free boson. In 5.1, we construct the free boson on the torus. In 5.2, we construct the orbifold of the free boson by a group of translations. Finally, in 5.3, we construct the orbifold of the free boson by $\mathbb{Z}_2$.

5.1 The boson on the torus

We have seen that any orbifold construction must first be projected onto the torus, and that it is therefore helpful to consider the spacetime to also be a torus. We would therefore like to use our results so far to calculate the partition function Z of the free boson on a torus.

Let's try to compute the partition function in a roundabout way. Recall what we have seen in 1.2.4. We call $\chi_{c,h}(\tau)$ the character of a Verma module, defined by

$$\begin{aligned}\chi_{c,h}(\tau) &= \mathrm{Tr}(q^{L_0-\frac{c}{24}}) \qquad \text{with} \quad q \equiv e^{2\pi i\tau} \\ &= \sum_{n=0}^{\infty} \dim(h+n) q^{n+h-\frac{c}{24}}\end{aligned} \tag{5.1}$$

where $\dim(h+n)$ is the number of linearly independent states of the Verma module $V(c,h)$ at level n. We can induce from (1.31) that for a generic Verma module, $\dim(h+n)$ is equal to $p(n)$ the number of partitions of $\mathbb{N}_n$. Knowing the generating function of $p(n)$

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(n)q^n \tag{5.2}$$

we can rewrite (5.1) in terms of the Dedekin's η function

$$\chi_{c,h}(\tau) = \frac{q^{h+\frac{1-c}{24}}}{\eta(\tau)} \tag{5.3}$$

Based on this, recalling the formula for the partition function (4.43), we can expect the partition function of the free boson to behave as

$$Z(\tau) \propto \frac{1}{|\eta(\tau)|^2} \tag{5.4}$$

The proportionality constant is important, as $\frac{1}{|\eta(\tau)|^2}$ is not modular invariant. After a quick computation, we can show that to make $|\eta(\tau)|$ invariant, we need to multiply it by $(\text{Im}(\tau))^{\frac{1}{4}}$. Thus, we can expect the partition function to take the form

$$Z(\tau) = \frac{1}{(\text{Im}(\tau))^{\frac{1}{2}}|\eta(\tau)|^2} \tag{5.5}$$

In fact, computing explicitely $Z(\tau)$ from its explicit formula (2.11) with the action of the boson given in (1.33) using complex analysis, we find exactly

$$Z(\tau) = \frac{1}{(\text{Im}(\tau))^{\frac{1}{2}}|\eta(\tau)|^2} \tag{5.6}$$

The details of this explicit computation can be found here [FMS97, p. 341].

The next two sections will display two exemples of orbifolds construction, by computing the partition function of two new theories obtained by quotienting the theory of the free boson on a torus by two different spatial groups.

5.2 The compactified boson

The Lagrangian of the boson (1.33) is invariant through translation $\varphi \to \varphi + C$. We can thus try to define the field φ on a circle of radius R hoping that it doesn't modify much the dynamics of the system. This is done by identifying φ with $\varphi + 2\pi R$, or equivalently by supposing that the field φ is invariant under symmetries of the form $\varphi \to \varphi + 2n\pi R$. This theory is usually called the compactified boson. Defining S as the space group generated by translations of $2\pi R$, we are thus trying to compute the orbifold of the usual free boson by the space group S. This has 2 main consequences:

- For the vertex operators to be well defined, the momentum operator π_0 can no longer take any value, and must be an integer of $\frac{1}{R}$.
- As described by the section 4.2.2, an infinite number of winding sectors are created, each associated with a different boundary condition.

If we consider the free boson on a cylinder of circumference L, we can associate to each winding sector an integer $m \in \mathbb{Z}$ called the winding number, such as the boundary condition for fields in the sector is

$$\varphi(x + L, t) = \varphi(x, t) + 2\pi m R \tag{5.7}$$

Considering the free boson on the torus defined by the periods ω_1, ω_2, windings can occur in 2 different directions: when going from z to $z + \omega_1$, and when going from z to $z + \omega_2$. The boundary conditions (and the winding sectors) are therefore parameterized by two integers $(m, m') \in \mathbb{Z}$:

$$\varphi(z + k\omega_1 + k'\omega_2) = \phi(z) + 2\pi R(km + k'm') \tag{5.8}$$

We can define the partition function $Z_{m,m'}$ as the partition function of fields in the sector (m, m').

Just as one would solve a differential equation using the homogeneous solution and a special solution, we can decompose our field φ into a special solution $\varphi^{cl}_{m,m'}$, solution to the classical equation of motion, and a periodic field $\tilde{\varphi}$, the "free part" of φ. We have

$$\varphi = \varphi^{cl}_{m,m'} + \tilde{\varphi}\varphi^{cl}_{m,m'} = 2\pi R\left(\frac{z}{\omega_1}\frac{m\bar{\tau}-m'}{t\bar{a}u-\tau} - \frac{\bar{z}}{\omega_1^*}\frac{m\tau-m'}{t\bar{a}u-\tau}\right) \tag{5.9}$$

We can easily check that $\varphi^{cl}_{m,m'}$ is real, and indeed verifies the boundary condition (5.8). We have $\Delta\varphi^{cl}_{m,m'} = 0$ and

$$\int d^2x\ \nabla\varphi^{cl}_{m,m'}\nabla\tilde{\varphi} = -\int d^2x\tilde{\varphi}\Delta\varphi^{cl}_{m,m'} = 0 \tag{5.10}$$

so recalling the action given of the field given by (1.33), the action $S[\varphi]$ decouples into $\S[\tilde{\varphi}]$ and $S[\varphi^{cl}_{m,m'}]$. We can easily compute $S[\varphi^{cl}_{m,m'}]$:

$$\begin{aligned} S[\varphi^{cl}_{m,m'}] &= \frac{1}{8^pi}\int d^2x\ (\nabla\varphi^{cl}_{m,m'})^2 \\ &= \frac{1}{2\pi}\int dzd\bar{z}\ \partial\varphi^{cl}_{m,m'}\bar{\partial}\varphi^{cl}_{m,m'} \\ &= 2\pi R^2\text{Im}(\omega_2\omega_1^*)\frac{1}{|\omega_1|^2}\left|\frac{m\tau-m'}{\tau-\bar{\tau}}\right|^2 \\ &= \pi R^2\frac{|m\tau-m'|^2}{2\text{Im}(\tau)} \end{aligned} \tag{5.11}$$

We write Z_{bos} the partition function of the boson on a torus, given by (5.6). Integrating according to the formula for Z given in (2.11) the "free part" of the field results in Z_{bos}. We therefore have

$$Z_{m,m'}(\tau) = Z_{\text{bos}}(\tau)e^{-\frac{\pi R^2|m\tau-m'|^2}{2\text{Im}(\tau)}} \tag{5.12}$$

Summing the partition function of all possible winding sectors, we get the partition function of the theory

$$Z(R) = \frac{R}{\sqrt{2}}Z_{\text{bos}}(\tau)\sum_{m,m'}e^{-\frac{\pi R^2|m\tau-m'|^2}{2\text{Im}(\tau)}} \tag{5.13}$$

which is modular invariant. Using Poisson's resummation, we get

$$Z(R) = \frac{1}{|\eta(\tau)|^2}\sum_{e,m\in\mathbb{Z}} q^{(\frac{e}{R}+\frac{mR}{2})^2/2}\bar{q}^{(\frac{e}{R}+\frac{mR}{2})^2/2} \tag{5.14}$$

We do not go into the details of this computation, which can be found here [FMS97, p. 351]

5.3 The $\mathbb{Z}_2$ orbifold

We now want to consider another kind of symmetry added to our system. We suppose again that the field φ has values in the circle, but we identify φ to

$-\varphi$, such that the resulting space for φ is a segment. We recognize here the $\mathbb{Z}_2$ orbifold shown as an exemple in 4.2.2, where the elements of the spacegroup are of the form $(\pm 1, n)$ for $n \in \mathbb{Z}$. Let's consider the orbifold made of the free boson on the torus quotiented by this spacegroup. We still have the winding sectors (m, m') as for the compact boson. But as seen in 4.2.2, we now also have the possibility of twisted sectors, associated with the following twisted boundary conditions

$$\begin{aligned} \varphi(z + k\omega_1 + k'\omega_2) &= e^{\pi i k}\varphi \\ \varphi(z + k\omega_1 + k'\omega_2) &= e^{\pi i k'}\varphi \\ \varphi(z + k\omega_1 + k'\omega_2) &= e^{\pi i (k+k')}\varphi \end{aligned} \tag{5.15}$$

Once again, we get more different possibilities of boundary conditions, as on the torus there are two ways of looping from z back to z: $z \to z+\omega_1$ and $z \to z+\omega_2$. We consider the general boundary condition parametrized by (u, v):

$$\varphi(z + k\omega_1 + k'\omega_2) = e^{2\pi i (kv + k'u)}\varphi \tag{5.16}$$

with v, u each being 0 or $\frac{1}{2}$.

We call $Z_{v,u}$ the partition function associated to the sector (v, u). In particular, $Z_{0,0} = Z(R)$. As the holomorphic and antiholomorphic fields are decoupled, we can integrate only the holomorphic part in the definition (2.11) of the partition function, and writing the result $f_{v,u}$, we then have

$$Z_{v,u} = |f_{v,u}|^2 \tag{5.17}$$

We want to compute $f_{v,u}$. To do so, we will use a method presented in [Hub13]. We will not go through every details, but rather present the general tricks. We want to compute the partition function in the operator formalism. However, we need to implement the boundary condition. For a boundary condition $v = 0$ or $u = 0$, nothing needs to be done. In fact, upon integration, the fields inside the correlation function commute due to the time-ordering, but commutating 2 fields has no impact on the correlation function as the field is bosonic. To implement an antiperiodic boundary condition, we need to implement the operator G such that $G\varphi G^{-1} = -\varphi$. Using it, we can calculate

$$f_{0,\frac{1}{2}} = \mathrm{Tr}(Gq^{L_0 - \frac{1}{24}}) \qquad f_{\frac{1}{2},0} = \mathrm{Tr}(q^{L_0 - \frac{1}{48}}) \qquad f_{\frac{1}{2},\frac{1}{2}} = \mathrm{Tr}(Gq^{L_0 - \frac{1}{48}}) \tag{5.18}$$

With (3.44), we can further develop these expressions

$$\begin{aligned} f_{0,\frac{1}{2}} &= \mathrm{Tr}(Gq^{\sum_n a_{-n}a_n - \frac{1}{24}}) \\ f_{\frac{1}{2},0} &= \mathrm{Tr}(q^{\sum_n a_{-n}a_n - \frac{1}{48}}) \\ f_{\frac{1}{2},\frac{1}{2}} &= \mathrm{Tr}(Gq^{\sum_n a_{-n}a_n - \frac{1}{48}}) \end{aligned} \tag{5.19}$$

But we know that G acts on vacua $|m, n\rangle$ as

$$G|m, n\rangle = |-m, -n\rangle \tag{5.20}$$

and that G anticommutes with the $(a_n)_n$. We can then see that due to the action of G, most of the terms in the expression of $f_{0,\frac{1}{2}}$ in (5.19) cancels out, and that only the states descending from the vacuum $|0,0\rangle$ has a contribution to the trace. Similarly, we see that in the expression of $f_{\frac{1}{2},\frac{1}{2}}$, only the vacua $|0,0\rangle_{pm}$ contribute. Putting everything together, we get

$$f_{0,\frac{1}{2}} = \sqrt{\frac{2\eta(\tau)}{\Theta_2(\tau)}} \quad f_{\frac{1}{2},0} = \sqrt{\frac{2\eta(\tau)}{\Theta_4(\tau)}} f_{\frac{1}{2},\frac{1}{2}} = \sqrt{\frac{2\eta(\tau)}{\Theta_3(\tau)}} \tag{5.21}$$

where the thetas are defined in (4.31).

With the help of (4.32), we can compute how these quantities transform under the modular group. We deduce that the only two quantities that are modular invariant are $Z_{0,0} = Z(R)$ and

$$\left|f_{0,\frac{1}{2}}\right|^2 + \left|f_{\frac{1}{2},0}\right|^2 + \left|f_{\frac{1}{2},\frac{1}{2}}\right|^2 \tag{5.22}$$

We get the total partition function of the theory by summing the partition function of all types of boundary conditions, and projecting everything on G-invariant states, as described in 4.2.1. We get

$$Z_{\text{orb}}(R) = \frac{1}{2}Z(R) + \frac{|\eta|}{|\Theta_2|} + \frac{|\eta|}{|\Theta_3|} + \frac{|\eta|}{|\Theta_4|} \tag{5.23}$$

Using the identity (4.35), we finally get

$$Z_{\text{orb}}(R) = \frac{1}{2}\left(Z(R) + \frac{|\Theta_2\Theta_3|}{|\eta|^2} + \frac{|\Theta_2\Theta 43|}{|\eta|^2} + \frac{|\Theta_3\Theta_4|}{|\eta|^2}\right) \tag{5.24}$$

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