First part of the notes for 'Physics for W-algebraists'

Buisine Léo
Seoul National University

May 14, 2024

Abstract

These notes were made in preparation for the first part of the seminar given by myself on physics for W-algebraists, in the team seminar of Pr. Suh UhiRinn at Seoul National University the 2024/05/14. In these, I introduce quantum mechanics, quantum field theories and conformal field theories for non physicists, by focusing on the intuitive side. In particular, I try to explain the reason for a lot of formulas, relations and operators appearing in mathematical physics, I also give the tools necessary to later understand the WZW-model and its reductions. I end these notes by showcasing a simple exemple of 2D CFT, the free boson. Be aware that these notes weren't proofread by anyone, and that there are most probably errors in it. One should not have too much faith in the details of every formula. However, the intuition behind each of them should be right.

Contents

Chapter 1

Quantum mechanics

The main idea of quantum mechanics, compared to classical mechanics, is to add some form of uncertainty to the data of a system. Instead of considering measurable quantities such as position or momentum, we consider probability densities on the space of these quantities, such as probability densities on the space of all possible positions and momentums. This leads to the intepretation of particles as "wave" as often heard when speaking of quantum mechanics, where the "wave" is the probability density.

1.1 A first toy model

To better understand the idea of quantum mechanics, let's first consider a simple massive system in 1 spatial dimension, whose physical state at a given time is defined solely by its position and derivatives according to time. We can for exemple imagine it to be a particle, or a bowling ball.

What do classical mechanics say about this sytem? We can write the position of the object according to time $x(t)$, its velocity $\dot{x}(t)$, and its acceleration $\ddot{x}(t)$. Given the forces $F(t)$ acting on the object, Newton's first law then says

$$\vec{F}(t) = m\ddot{x}(t) \tag{1.1}$$

Supposing $F(t)$ is a known external parameter, $\ddot{x}(t)$ and higher derivatives of $x(t)$ are completely determined. For this reason, we only consider the position and velocity of the object, linked by the relation

$$\dot{x}(t) = \frac{\mathrm{d}x}{\mathrm{d}t}(t) \tag{1.2}$$

Given a starting position and starting velocity, the dynamics of the system are completely known, determined by

$$x(t) = x(0) + t\dot{x}(0) + C(F,t) \tag{1.3}$$

With

$$C(F,t) = \int_0^t \int_0^T F(\lambda)\mathrm{d}T\mathrm{d}\lambda \tag{1.4}$$

From this, how can we integrate the idea of quantum mechanics? We would like to introduce some uncertainty in our system. Moreover, quantum mechanics say that some quantities are linked such that the more we know of one, the less we know of the other. In our exemple, velocity and position are linked, such that the more we know of the position, the less we know of the velocity, and reciprocally.

To picture these, imagine a bowling ball rolling on a line[1]. We want to measure the velocity and position of the ball, but we only have a cheap and slow camera. We may take one photo with it, or a serie of photos with somewhat regular time interval. If we take one single photo, we might have a good idea (though not perfectly accurate due to the slowness of the camera) of the position of the ball, but we won't have any idea of its velocity. If we take a serie of photos, we might have a good guess of its velocity, but we won't know actually where is the ball among the different pictures.

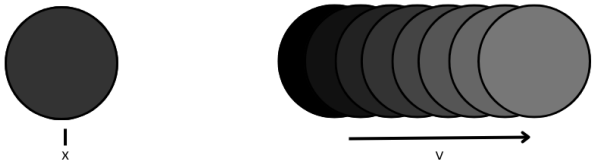

Figure 1.1: Pictures of a bowling ball rolling

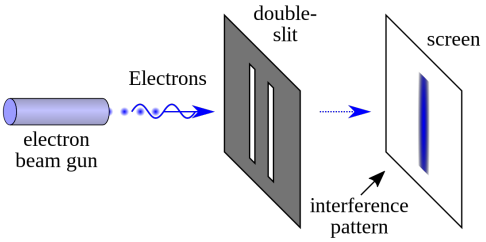

Figure 1.2: The double slit experiment

Remark. Although this comparison makes it easier to understand the uncertainty of quantum mechanics, it should remain a simple comparison. In quantum mechanics, the uncertainty is inherent to the system, and not due to cheap measure instruments. A particle is no more embodied by a position but by a probability measure on all possible positions. In fact, this idea of uncertainty was developped after physicists realized electrons sometimes act as waves and can interfere with themselves, as shown by the double-slit experiment (fig.1.2). The "wave" is interpreted to be the probability wave, showing the intrinsic nature of the uncertainty in quantum mechanics.

[1] This comparison was given to me by my little brother

Now, how can one capture these ideas in a framework? Recall that in classical mechanics, the data is captured by 2 functions

$$\begin{aligned} x &: \mathbb{R} \to \mathbb{R} \\ \dot{x} &: \mathbb{R} \to \mathbb{R} \end{aligned} \tag{1.5}$$

which are in fact redundant, and linked by the relation (1.2). Replacing the data of a known position or velocity by a probability on the space of all possible positions or velocities, we may have

$$\begin{aligned} X &: \mathbb{R} \to (\mathbb{R} \to \mathbb{R}_+) \\ V &: \mathbb{R} \to (\mathbb{R} \to \mathbb{R}_+) \end{aligned} \tag{1.6}$$

However, transforming the relation (1.2) isn't as straightforward. If we simply had a similar relation, the uncertainty of the position would be proportional to the one of the velocity, instead of being inversely proportional.

The right idea, first introced by Heisenberg, Born and Jordan as an axiom called the "quantum condition", is to use some kind of Fourier transform. This use of the Fourier transform will be better justified later, but we should now give the intuition behind it. Classically, the Fourier transform is used to decompose a periodic signal onto a basis of standard signals with different frequencies. In our case, we may understand the "frequency" of a probability density as the velocity of the object studied, and assimilate the basis of standard signals with different frequencies as a basis of standard objects with different velocities. Then, as one can understand by looking at figure 1.1, an object with very well defined position won't have a very well defined velocity, whilst an object with very well defined velocity won't have a very well defined position.

Formally, to introduce the Fourier transform, we should allow for one more degree of freedom in the probability functions, and replace the data suggested in (1.6) by

$$\begin{aligned} X &: \mathbb{R} \to (\mathbb{R} \to \mathbb{C}) \\ V &: \mathbb{R} \to (\mathbb{R} \to \mathbb{C}) \end{aligned} \tag{1.7}$$

where we read the probability by taking the norm on $\mathbb{C}$. Then, we may link the two functions at a given time by

$$\begin{aligned} V(t)(v) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ivx} X(t)(x) \mathrm{d}x \\ X(t)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixv} V(t)(v) \mathrm{d}v \end{aligned} \tag{1.8}$$

Note how in this framework, an object with a known position would be represented by a delta function in the position space and by a forever rotating function with diverging integral in the velocity space, and reciprocally for an object with known velocity.

1.2 The general framework

How do one generalise this to any kind of system? In general, we may have any space of configuration instead of simply $\mathbb{R}$ for the space of all possible positions.

We want to consider functions from a space of configurations to $\mathbb{C}$. But we may instead treat the space of configurations as a vector space (where each dimension corresponds to a configuration) and consider complex forms on this vector space, or equivalently complex vectors in this space. This leads to the first axiom

1. All possible states of the physical system are described by a vector in a separable complex Hilbert space $\mathcal{H}$

Next, we want to be able to read off data from vectors in $\mathcal{H}$. Basically, we want to give meaning to the dimensions in $\mathcal{H}$. This is done by associating to each measurable value an operator, whose eigenvalues are the possibles values, and eigenvectors are the proper states having these values. For exemple, in our toy model, we would have an operator giving the position, having all of $\mathbb{R}$ as eigenvalues, with the eigenvector $\delta(x-y)$ associated to the eigenvalue y, in the position space. In term of axioms, this reads

2. To each observable physical property $\mathcal{O}$ corresponds a self-adjoint operator A on $\mathcal{H}$

3. If A is the operator of an observable $\mathcal{O}$ then any experimentally observed value of $\mathcal{O}$ must be an eigenvalue of A.

4. For any observable $\mathcal{O}$, with associated operator A, whose eigenvectors and eigenvalues are $\{x_n, \lambda_n\}_n$, given a system in the state $\psi \in \mathcal{H}$, the probability for this system to measure λ for the observable $\mathcal{O}$ is equal to

$$\mathbb{P}(\lambda) = \frac{\sum_{i:\lambda_i=\lambda} \langle x_i, \psi\rangle}{\langle \psi, \psi\rangle} \tag{1.9}$$

At a first glance, (1.9) doesn't seem very convenient to use. We thus have some notations and conventions to make this more usable.

First of all, we usually suppose the state of the system to be normalised, that is $\langle \psi, \psi\rangle = 1$. With this, (1.9) reduces to

$$\mathbb{P}(\lambda) = \sum_{i:\lambda_i=\lambda} \langle x_i, \psi\rangle \tag{1.10}$$

We also use Dirac's notations. Dirac's idea was simply to have better notations to deal with brackets, since most of quantum mechanics computations come down to brackets. The bracket notation doesn't intuitively show its linearity, and doesn't allow for an easy expression of operator. Therefore, Dirac suggested to break down the bracket $\langle \psi, \phi\rangle$ into two parts: the bra $\langle \psi|$, and the ket $|\phi\rangle$. In these notations, $|\phi\rangle \in \mathcal{H}$ simply corresponds to the vector ϕ, whilst $\langle \psi| \in \mathcal{H}^*$ corresponds to the form dual to ψ. This way, we obviously have $\langle \psi|\phi\rangle \equiv \langle \psi||\phi\rangle = \langle \psi, \phi\rangle$, but we can also think of the two as separate entities. For exemple, linearity now simply correspond to distributivity in the computations

$$\langle \psi| \left(\sum_i |\phi_i\rangle \right) \tag{1.11}$$

and

$$\left(\sum_i \langle\psi_i|\right)|\phi\rangle \tag{1.12}$$

In these notations, we can also easily measure physical quantities. For exemple, for a given observable $\mathcal{O}$ whose associated operator is A with eigenvalues and eigenvectors $\{x_n, \lambda_n\}_n$, one would typically write

$$\langle\lambda| \equiv \sum_{i:\lambda_i=\lambda} \langle x_i| \tag{1.13}$$

Then, (1.9) becomes

$$\mathbb{P}(\lambda) = \langle\lambda|\psi\rangle \tag{1.14}$$

With these notations, we can also define operators as kets followed by bras. In particular, with the notations from the exemple above, we can easily define the projector onto the eigenspace associated to the eigenvalue λ

$$P_\lambda = |\lambda\rangle\langle\lambda| \tag{1.15}$$

Given $(\lambda_k)_k$ the different measurable values of $\mathcal{O}$, we then have

$$\sum_k |\lambda_k\rangle\langle\lambda_k| = 1 \tag{1.16}$$

where here 1 is the identity on $\mathcal{H}$.

1.3 The Hamiltonian

Let's now try to generalize (1.8) to our framework. Actually, this may be done very naturally.

Our system should be invariant through time translation: if a system act a certain way now, it should act the same way later on. Therefore, a time symmetry is present in the system, and there should be a representation of this symmetry on the Hilbert space $\mathcal{H}$. The representation of a symmetry is either unitary or antiunitary. We may assume it is unitary. Moreover, the symmetry is parametrized by time $\mathbb{R}$, which is continuous. We may therefore suppose that the image of $\mathbb{R}$ on the operators acting on $\mathcal{H}$ through the representation constitutes a strongly continous unitary group, meaning if we write this representation $(U(t))_{t\in\mathbb{R}}$, we may suppose that the following hold true:

- $\forall s, t \in \mathbb{R}, U(t+s) = U(s)U(t)$
- for any $x \in \mathcal{H}$ the map $t \to U(t)x$ is continuous

This should also be an axiom of quantum mechanics, as we assumed unitarity and strong continuity:

5. If the system is not affected by external influences, then its state evolves in time as $|\psi(t)\rangle = U(t)|\psi\rangle$ for some strongly unitary group U that solely depends on the system

From this, we may use Stone's theorem:

Theorem (Stone's theorem)**.** *To any strongly continuous unitary group U, we can associate a unique self-adjoint operator H. We then write $U(t) = e^{-\mathrm{i}tH}$.*

Supposing our system is not affected by external influences (we should add any external force into our system), we then have a self-adjoint operator H, such that

$$|\psi(t)\rangle = e^{-\mathrm{i}tH}|\psi\rangle \tag{1.17}$$

But

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{-\mathrm{i}tH} = -\mathrm{i}He^{-\mathrm{i}tH} \tag{1.18}$$

such that (1.17) may be rewritten

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H|\psi(t)\rangle \tag{1.19}$$

This equation is known as Schrödinger's equation, and governs the time evolution of the state $|\psi(t)\rangle$

In the right-hand side of Schrödinger's equation, we can see the operator H given by Stone's theorem appear in fron of our state. What is its meaning? If $|\psi(t)\rangle$ is an eigenvector of H, we may define E to be its associated eigenvalue. Then, the rate at which $|\psi(t)\rangle$ changes is proportional to E. The bigger E is, the more $|\psi(t)\rangle$ will move between configurations, whilst if E is equal to 0 then $|\psi(t)\rangle$ will stay constant. We can therefore interpret E as the energy, and H as the operator giving the energy. As a direct comparison to classical mechanics, we call H the *Hamiltonian*, as in Hamiltonian mechanics.

Remark. Usually, for a free particle, we take $H = \frac{P^2}{2m}$ to be the kinetic energy, where m is the mass of the particle and P the operator giving its momentum. Writing X the position operator, Schrödinger's equation leads to the commutation relation

$$[X, P] = \mathrm{i} \tag{1.20}$$

which, in turn, necessarily leads to (1.8) when considering a particle moving in 1 dimension.

1.4 System with multiple particles

In our toy model, we considered a system made of one particle. From this, how can one handle multiple particles? And how can one construct a system with a variable number of particles? The key here is to consider the system as a whole, instead of just considering a system made up of one particle.

For the following discussion, we will only talk of bosons. Bosons are the most basic particles one could imagine. In particular, they are indistinguishable one from another. Suppose we have $\mathcal{H}$, the Hilbert space capable of describing all possible configurations of a single boson. Naively, we may want to use $\mathcal{H}^{\oplus n}$, the product of n copies of $\mathcal{H}$, to model n bosons. However, in that case, one could number each particle, and they would therefore be distinguishable. To bypass this issue, we must symmetrize this space. The final product should then be $\mathcal{H}_{\mathrm{sym}}^{\oplus n}$, whose complete construction won't be shown here. However, it can be found in [Cha18].

Now that we have constructed the space of configuration for any number of particles, we should try to construct a space $\mathcal{B}$ capable of modelling a variable number of particles. This can be made by taking the sum of all $\mathcal{H}_{\text{sym}}^{\oplus n}$, but by forcing the states in it to have a finite norm. Formally, we have

$$\mathcal{B} = \left\{ (\psi_0, \psi_1, \dots) \in \bigoplus_{n\in\mathbb{N}} \mathcal{H}_{\text{sym}}^{\oplus n} \quad / \quad \sum_{n=0}^{\infty} \langle \psi_n | \psi_n \rangle < \infty \right\} \tag{1.21}$$

This space capable of holding any number of particles is called the *Fock space*. In our case, it is the bosonic Fock space.

Now, let (e_k) be a basis of $\mathcal{H}$. We have a cannonical basis on the associated Fock space. For a state having n_k bosons in the state e_k for all k, we write: $|n_1, n_2, n_3, \dots\rangle$. Defining this basis on the Fock space and reformulating everything in it is called the *second quantization*. From there, we can easily create the creation and annihilation operators.

For k, we define the creation operator $a_k^\dagger : \mathcal{B} \to \mathcal{B}$ that create a boson in state e_k, as follow:

$$a_k^\dagger |n_1, n_2, \dots\rangle = \sqrt{n_k + 1} |n_1, n_2, \dots, n_{k-1}, n_k + 1, n_{k+1}, \dots\rangle \tag{1.22}$$

We similarly define the annihilation operator $a_k : \mathcal{B} \to \mathcal{B}$ that destroy a boson in state e_k, as follow:

$$\begin{aligned} a_k |n_1, n_2, \dots\rangle &= \sqrt{n_k} |n_1, n_2, \dots, n_{k-1}, n_k - 1, n_{k+1}, \dots\rangle \text{ if } n_k > 0 \\ &= 0 \text{ if } n_k = 0 \end{aligned} \tag{1.23}$$

We can also define the density operator $a_k^\dagger a_k$, which acts on states as

$$a_k^\dagger a_k |n_1, n_2, \dots\rangle = n_k |n_1, n_2, \dots\rangle \tag{1.24}$$

When the context is clear, we may sometimes write this operator $n_k \equiv a_k^\dagger a_k$.

Remark. Note how in the definition of the density operator, we should have the annihilation operator before the creation operator. This order makes sure the operator returns a 0 when the state it is acting on is empty, or more generally if there is no particle in the state e_k. Taking the operators the other way around, we would get a value of 1 upon acting on the vacuum state. This will have its importance later on, and will motivate normal ordering.

1.5 Changing the point of view

So far, we have always considered a state evolving in time, upon which fixed operators act. This point of view is called the Schrödinger picture, but it is not the only one. We will finish this section by looking at another point of view called Heisenberg picture, which will prove useful in the following section.

Up until now, we have considered the state of a system, for exemple a particle, which is evolving in time according to the relation

$$|\psi(t)\rangle = e^{-itH} |\psi(0)\rangle \tag{1.25}$$

In this setting, the world is fixed, and the particle is moving in it. But we can also switch to the point of view of the particle, from which the particle itself is fixed and the world is moving. Translating this into the quantum framework, this means we could also consider a fixed $|\psi\rangle$ describing the particle across all spacetime, and have the operators depend on time. This is the so-called Heisenberg picture. In this picture, the operators evolve instead of the state. Instead of (1.25), for an operator A associated to an observable $\mathcal{O}$, we would have

$$A(t) = e^{itH} A(0) e^{-itH} \tag{1.26}$$

In the case of a single particle for exemple, both the position operator X and momentum operator P would have a dependance in time. At a fixed time t_1, making $X(t_1)$ act on $|\psi\rangle$ would then return the position of the particle at t_1.

We should emphasize that these two pictures describe the same dynamics. When evaluating the expected value of the physical quantity $\mathcal{O}$ of a system $|\psi(t)\rangle$, physical quantity whose associated operator is A, we have

$$\begin{aligned} \langle \psi(t)|A|\psi(t)\rangle &= \langle e^{-itH}\psi|A|e^{-itH}\psi\rangle \\ &= \langle \psi|e^{itH}Ae^{-itH}|\psi\rangle \\ &= \langle \psi|A(t)|\psi\rangle \end{aligned} \tag{1.27}$$

However, it will be more convenient to think in the Heisenberg picture for quantum field theories.

Chapter 2

Quantum field theory

Up until now, we have worked in a framework able to describe particles using probability functions, as to give them wave behaviours. In quantum mechanics, a "state" is the data of a vector in a complex configuration space, describing the probability of the system to be in each of the possible configurations at each given time. But at this point, it might seem artificial to keep a strict notion of particle, and we might as well describe the system by a non-normalized vector giving the probability to find "something" in each of the possible configurations at each given time. This is the idea behind quantum field theories.

2.1 Wightman's axioms

In practice, how does one define quantum field theories (QFTs)? There are still a few modifications to do from quantum mechanics to reach this new framework.

We still consider an Hilbert space $\mathcal{H}$ able to describe any configuration of the system. But now, the system won't have a probability of being in each of the configurations given by $\mathcal{H}$. Instead, there will be some probability to find a part of the system in each of the configurations given by $\mathcal{H}$. The key difference is to interpret $\mathcal{H}$ as a space in which the system lives instead of as a space of different possible configurations.

Another difference is that we will always want QFTs to be consistent with other areas of physics,and in particular special relativity. It is possible to make quantum mechanics consistent with special relativity, but not straightforward. In QFTs, we will enforce the notion of causality (nothing can travel faster than light), and we will enforce the symmetries given by special relativity, that is the Poincaré invariance. The Poincaré group is defined as the sum of the group of translations and of the Lorentz group, made up of rotations and boosts.

But to enforce the Poincaré invariance, we will need to have a notion of space, homogeneous with the notion of time. To have this, we stay close to the Heisenberg picture from quantum mechanics. We consider states $|*\rangle$ describing the state of the system across all of spacetime, and we give the operators a dependance in time, and space. This therefore turn the operators in fields of operators, explaining the name "quantum field theory".

Moreover, to have a better defined framework, and to unify the data with the probability densities on $\mathcal{H}$, instead of giving the operators a straightforward dependance on spacetime, we give them a dependance on test functions on spacetime. In other words, we upgrade our operators from fields of operators on spacetime to distributions of operators to spacetime. Nethertheless, we still call them fields, as was done historically.

Finally, now that the focus has moved from states to operators, we should still make sure we have a priviledged state to work on. We therefore require any QFT to have a well defined vacuum state, representing the absence of anything, the true vacuum, in $\mathcal{H}$.

Now that we have seen most of the ideas of QFT, we should give a proper mathematical definition. Wightman's axiom are the most common axioms for mathematically rigourously defining quantum field theories.

Definition 2.1.1. Let M be the spacetime of our theory, usually the d dimensional Minkowski space. An Hilbert space $\mathcal{H}$ and a collection of operator-valued distributions on M (φ_a) form a *quantum field theory* if

1. We have a unitary representation $U : (q, \Lambda) \to U(q, \Lambda)$ of the Poincaré group such that for all $q \in M, \Lambda \in L$ the Lorentz group, for all a, f, we have
$$U(q, \Lambda)\varphi_a(f)U(q, \Lambda)^{-1} = \varphi_a((q, \Lambda)f) \tag{2.1}$$
2. We have a vacuum vector $|0\rangle$ fixed by U, and in the domain of any polynomial having as indeterminates the (φ_a)
3. For $q \in M$, writing $U(q, 1) = e^{i\sum_{k=0}^{d-1} q_k P_k}$, the joint spectrum of the P_k lies in the forward cone
4. The linear subspace of the polynomials having as indeterminates the (φ_a) applied to $|0\rangle$ is dense in $\mathcal{H}$. In other words, $\{P|0\rangle / P \in \mathbb{P}((\varphi_a))\}$ is dense in $\mathcal{H}$
5. For any test function f, g whose support are spacelike separated (any two point is spacelike separated), for any a, b, we have $[\varphi_a(f), \varphi_b(g)] = 0$

The first axiom ensures Poincaré invariance. Note how (2.1) is nothing but a generalisation to the Poincaré group of the covariance through time translations, given by (1.26). The second axiom gives the vacuum state, and ensures it is stable through the symmetries (as it should be, since the vacuum state describes a world devoid of anything). Moreover, it makes sure we can apply any field to it. The third axiom is a kind on unitarity condition. It makes sure the light cone in M corresponds to an idea of light cone in $\mathcal{H}$. The fourth axiom makes sure that all states described by $\mathcal{H}$ are physical, meaning we can reach almost any of them using the fields given by the theory, starting from the vacuum. Sometimes, this axiom appear in an even stricter version, requiring all states in $\mathcal{H}$ to be reachable using fields from the vacuum. Finally, the last axiom makes sure causality is transmitted from M to $\mathcal{H}$, and prevents operators coming from spacelike separated tempered functions to interact which each other.

In QFT, we will more often consider fields than states, as (almost) any state can be found by applying a field on the vacuum. For this reason, we will often consider a field φ, and write its corresponding state $|\varphi\rangle \equiv \varphi|0\rangle$.

2.2 Correlation functions

Now that we have understood the framework with which we will be working on, we want to introduce the main object of study. In general, our main goal when studying a QFT is to compute its correlation functions. To understand these objects and their importance, let's first try to understand it in the classical case.

We place ourselves in classical mechanics. Let's consider a system made up of a single particle, in presence of a potential V, vanishing at infinity. In other words, the particle is asymptotically free, but is affected by V non-asympotically. To better understand such a system, we may imagine for exemple the case of a meteorite moving near a planet. Far away from the planet, in the far past and far future, the meteorite won't care about the planet. In such a system, we want to know how the presence of a planet modifies the trajectory of the meteorite, meaning we want to know the position and direction the meteorite will be having in the far future if it came from a certain position and direction in the far past.

Let (x, v) be the position and velocity of a particle at time 0. Without potential, this particle would be in $(x - tv, v)$ at time $-t$. We write $\Omega^-_{0,t}(x, v) = (x - tv, v)$. Reciprocally, knowing the position and velocity of a particle (x, v) at time 0, we can compute $\Omega^+_t(x, v)$ the position and velocity of the same particle under the potential V at time t. We then have $\lim_{t\to\infty} \Omega^+_t\Omega^-_{0,t}(x, v)$ the position and velocity of the particle coming from the "direction (x, v)", under the action of the potential V. But we can similarly compute the "direction" in which will end a particle starting at (x, v), under the action of the potential V, with $\lim_{t\to\infty} \Omega^-_{0,t}\Omega^+_t(x, v)$. Therefore, we can compute $\lim_{t\to\infty} \Omega^-_{0,t}\Omega^+_t\Omega^+_t\Omega^-_{0,t}(x, v)$, the direction in which the particle ends when it came from the direction (x, v). We call $S = \lim_{t\to\infty} \Omega^-_t\Omega^+_t\Omega^+_t\Omega^-_t$ the scattering operator. Using this operator, we may compute the far future state of a particle knowing its far past state, which is exactly what we would like to know.

By analogy, we can also use similar methods in quantum mechanics. With H the Hamiltonian of a system and H_0 the free Hamiltonian (the Hamiltonian of the system without external forces), we can define the evolution and free evolution operators. We have $\Omega^+_t = e^{-itH}$ and $\Omega^-_{0,t} = e^{itH_0}$. We can then define the scattering operator (under some conditions) and compute the probability of a particle coming from a direction ending in another direction

$$\mathbb{P}((x, v) \to (x', v')) = \langle x', v'|S|x, v\rangle \tag{2.2}$$

In the QFT framework, we may develop this even further. Using the creation $a^\dagger$ and annihilation a operator fields, we can write

$$\mathbb{P}((x, v) \to (x', v')) = \langle 0|a(x', v')Sa^\dagger(x, v)|0\rangle \tag{2.3}$$

More generally, given an operator $a^\dagger_0$ creating a particle e_0 and another operator a_1 destroying a particle e_1, the quantity

$$\langle 0|a_1Sa^\dagger_0|0\rangle \tag{2.4}$$

computes the probability for a particle e_0 entering the potential in a far past to become the particle e_1 in a far future.

We may not always simply have a single potential, or may not always want to compute asymptotic states. In a more general fashion, we will want to compute expressions of the form

$$\langle 0|\varphi_1(x_1)\varphi_2(x_2)\dots\varphi_n(x_n)|0\rangle \tag{2.5}$$

where $(\varphi_k)_k$ is a sequence of fields. However, this expression only makes sense if the operators are ordered according to time, meaning x_1 is later than x_2, which is later than x_3, etc... This leads to the definition of a correlation function.

Definition 2.2.1. Given a sequence of n fields $(\varphi_k)_{k\in\mathbb{N}_n}$, its *n-point correlation function*, or simply correlation function, is the function in n variables

$$\langle 0|\mathcal{T}(\varphi_1(x_1)\varphi_2(x_2)\dots\varphi_n(x_n))|0\rangle \tag{2.6}$$

where $\mathcal{T}$ is the time-ordering operator, which reorders the fields according to the time they are taken at.

For a given sequence of n fields $(\varphi_k)_{k\in\mathbb{N}_n}$, we write its correlation function $\langle\varphi_1(x_1)\varphi_2(x_2)\dots\varphi_n(x_n)\rangle$. Solving a QFT amounts, in general, to computing all of its correlation functions.

Remark. Note that the correlation function is not a real correlation function, in the sense of statistics or probability. In fact, the correlation function of two fields can be negative. However, it is the most correct generalisation of what we could imagine as a correlation function between fields.

Before ending this subsection, we will see our first tool to compute correlation function, the acclaimed Wick theorem.

Wick's theorem links the time ordering seen in correlation functions, to the normal ordering introduced in the remark following (1.24). As we began to explain in this remark, the normal ordering is commonly used to prevent getting values from the vacuum, and especially to prevent computing the vacuum's energy expectation, which often diverges. We define and write it as follow.

Definition 2.2.2. The *normal ordering* of a sequence of operators $\varphi_1\dots\varphi_n$, written $:\varphi_1\dots\varphi_n:$, is the product of these operators where any annihilation operator has been put to the right, and any creation operator has been put to the left.

For exemple, given a theory with a creation operator $a^\dagger$ and a annihilation operator a, $:aa^\dagger:=a^\dagger a$.

We also need to introduce contractions, which is the combinatoric tool used to express Wick's theorem.

Definition 2.2.3. Given a normal-ordered product of operators $:\varphi_1\dots\varphi_n:$, we define the *contraction* of the 2 operators φ_i and φ_j as the normal-ordered product $:\varphi_1\dots\varphi_n:$ where we removed φ_i and φ_j, multiplied by their correlation function $\langle\varphi_i\varphi_j\rangle$. We write it

$$:\varphi_1\dots\overbrace{\varphi_i\dots\varphi_j}\dots\varphi_n:$$

For exemple,

$$\begin{aligned} &: \varphi_1 \overbrace{\varphi_2 \varphi_3} \varphi_4 : \; =: \varphi_1 \varphi_3 : \langle \varphi_2 \varphi_4 \rangle \\ &: \overbrace{\varphi_1 \varphi_2 \varphi_3 \varphi_4} : \; = \langle \varphi_1 \varphi_3 \rangle \langle \varphi_2 \varphi_4 \rangle \end{aligned} \tag{2.7}$$

From these, Wick's theorem is pretty straightforward though very powerful. Its demonstration is not too hard but really tedious. We will not go through it, but it can be found in [FMS96].

Theorem (Wick's theorem)**.** *The time-ordered product is equal to the normal ordered product, plus all possible ways of contracting pairs within it.*

For exemple,

$$\begin{aligned} \mathcal{T}(\varphi_1\varphi_2\varphi_3\varphi_4) = \quad & : \varphi_1\varphi_2\varphi_3\varphi_4 : + \\ & : \overbrace{\varphi_1\varphi_2}\varphi_3\varphi_4 : + : \varphi_1 \overbrace{\varphi_2\varphi_3} \varphi_4 : + : \varphi_1\varphi_2 \overbrace{\varphi_3\varphi_4} : + \\ & : \overbrace{\varphi_1\varphi_2\varphi_3}\varphi_4 : + : \varphi_1 \overbrace{\varphi_2\varphi_3\varphi_4} : + : \overbrace{\varphi_1\varphi_2\varphi_3\varphi_4} : + \\ & : \overbrace{\varphi_1}\varphi_2 \overbrace{\varphi_3}\varphi_4 : + : \overbrace{\varphi_1 \overbrace{\varphi_2}\varphi_3\varphi_4} : + : \overbrace{\varphi_1 \overbrace{\varphi_2}\varphi_3 \overbrace{\varphi_4}} : \end{aligned} \tag{2.8}$$

This combinatoric theorem is elementary but very powerful. For exemple, it implies that the correlation function of any odd number of fields must always vanish.

It also implies that the correlation function of any number of fields can be reduced to propagators, that is 2-points correlation functions. For this reason, we will often only compute 2-points correlation functions, as any higher correlation function can be straightly found using them.

2.3 The path-integral formalism

Up until now, we have developped a framework using operator-valued distributions going from some space to another, which gives a very general but difficult to work with framework. We would like to reformulate everything in another way, to turn the landscape into something more familiar and easier to work with. To do so, we will inspire ourselves from Lagrangian mechanics and statistical mechanics.

We have multiple way to think about classical mechanics. In Newton's picture, one considers a system with some position, velocity and acceleration, on which acts some external forces, as discussed in 1.1. We then use Newton's first laws, which states that $\vec{F} = m\vec{a}$. But another way to picture thing is to look at the energy instead of the forces. In this case, we define the Lagrangian $\mathcal{L}$ which gives the energy of the system according to its state and derivatives, and we use Lagrange's principle which states that nature always try to minimise the energy spent, meaning the Lagrangian is always minimal along a trajectory. This statement results, more formally, in Euler-Lagrange equation, which is strictly equivalent to Newton's first law

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) = \frac{\partial \mathcal{L}}{\partial q_j} \tag{2.9}$$

From the Lagrangian, one can derive the action as the integral of the Lagrangian along the trajectory, corresponding to the energy needed for a system to go through this trajectory

$$\mathcal{S} = \int_{-\infty}^{\infty} \mathrm{d}t \ \mathcal{L} \tag{2.10}$$

Lagrange's principle then amounts to the fact that $\mathcal{S}$ should always be extremal, or in other words

$$\delta\mathcal{S} = 0 \tag{2.11}$$

under any small perturbation of the trajectory.

On the other hand, in statistical mechanics, the energetic point of view is particularly useful since the probability of finding the system in a specific state is linked to its energy. Let's imagine a system at temperature T, which can be in N differents states, where state i has energy E_i. Then Boltzmann's distribution says that the probability of finding the system in state i is equal to

$$\mathbb{P}_i = \frac{1}{Z} e^{-\frac{E_i}{kT}} \tag{2.12}$$

where Z is the partition function used to normalise this probability law

$$Z = \sum_{i=1}^{N} e^{-\frac{E_i}{kT}} \tag{2.13}$$

How can we use these theories to help us understand QFTs and compute correlation functions? First starting with quantum mechanics, we recall that we have defined an Hamiltonian, which should enable us to define a corresponding Lagrangian. But then, switching to Heisenberg's picture, recalling that a state describes the system across all of spacetime and therefore across all of its trajectory, we should be able to associate some sort of action to each state. This would then give us an energy associated to the state, potentially allowing us to find some equivalent to the Boltzmann distribution, linking correlation function and exponentials of this energy. To summarize, we expect to find some function $\mathcal{S}$ depending on the state such that we may link correlation functions to some normalized exponential of this function $\mathcal{S}$.

As suggested in the paragraph above, we start in the quantum mechanics framework. We suppose $\mathcal{H}$ to be parametrized by some q, whose derivative according to time is written p.[1] Given a state $\psi(t)$, we have

$$\langle q|\psi(T+t_0)\rangle = \langle q|e^{-iTH}|\psi(t_0)\rangle \tag{2.14}$$

We may divide this expontential into small bits, and insert the identity in between each exponentials

$$\langle q|\psi(T+t_0)\rangle = \langle q|e^{-iTH/N}1_1 e^{-iTH/N}1_2 \dots 1_{N-1}e^{-iTH/N}|\psi(t_0)\rangle \tag{2.15}$$

But we can replace the identities with complete basis, as

$$1_k = \int \mathrm{d}p_k |p_k\rangle\langle p_k| \int \mathrm{d}q_k |q_k\rangle\langle q_k| = \int \frac{\mathrm{d}p_k \mathrm{d}q_k}{\sqrt{2\pi}} e^{-\mathrm{i}p_k q_k} |p_k\rangle\langle q_k| \tag{2.16}$$

[1] These correspond to the canonical coordinates from Hamiltonian mechanics

Rewriting $q_N \equiv q$ for convenience, we have

$$\langle q|\psi(T+t_0)\rangle = \int \prod_{k=1}^{N-1} \frac{\mathrm{d}p_k \mathrm{d}q_k}{\sqrt{2\pi}} \left[\prod_{k=1}^{N-1} \langle q_{k+1}|e^{-iHT/N}|p_k\rangle e^{-\mathrm{i}p_k q_k}\right] \langle q_1|\psi(t_0)\rangle \tag{2.17}$$

But as $N \to \infty$, we may approximate

$$\langle q_{k+1}|e^{-iHT/N}|p_k\rangle = \frac{1}{\sqrt{2\pi}} e^{ip_k(q_{k+1}-q_k)} e^{-iH(p_k,q_k+1)T/N} \tag{2.18}$$

where $H(p,q)$ corresponds to the Hamiltonian on the system, that is $H|p,q\rangle = H(p,q)|p,q\rangle$

Inserting this back into (2.17) and taking the continuum limit as $N \to \infty$, we get

$$\langle q|\psi(T+t_0)\rangle = \int [DpDq] e^{i\int \mathrm{d}t(p\dot{q}-H(p,q))} \tag{2.19}$$

where $[DpDq]$ is the measure on the space of configuration taking into account all of the factors[2], and where the integration is made from the state $\psi(t_0)$ to the state q.

Someone used to Lagrangian mechanics and Hamiltonian mechanics would immediatly see that $p\dot{q} - H(p,q)$ exactly defines the usual Lagrangian $\mathcal{L}$, and therefore that $\int \mathrm{d}t(p\dot{q} - H(p,q))$ corresponds to the action of the field $\mathcal{S}(p,q)$. Further rewriting the canonical coordinates (p,q) as simply fields belonging to $\mathcal{H}$, we have in a very general way for any 2 states $|\psi_1\rangle, |\psi_2\rangle$ taken at two times t_1, t_2

$$\langle \psi_1(t_1)|\psi_2(t_2)\rangle = \int [d\psi] e^{i\mathcal{S}[\psi]} \tag{2.20}$$

where ψ interpolates between the states $|\psi_1\rangle$ and $|\psi_2\rangle$.

This can be easily adapted to QFTs, where we can find an action $\mathcal{S}$ and a measure on the space of fields $[d\varphi]$ such that we have for any fields φ_1, φ_2

$$\langle \varphi_1|\varphi_2\rangle = \int [d\varphi] e^{i\mathcal{S}[\varphi]} \tag{2.21}$$

This formalism is often called the path-integral formalism, as we integrate through the path taken by the state. Even though the way we introduced it might have seemed a bit artificial, some consider it more intrinsic to the framework than the operator formalism. Let's now try to reformulate correlation functions using this relation.

Let ψ be an arbitrary field. Writing $|n\rangle$ the energy eigenstates of the Hamiltonian with eigenvalue E_n, we have

$$\begin{aligned} e^{itH(1-i\epsilon)}|\psi\rangle &= \sum_n e^{itH(1-i\epsilon)}|n\rangle\langle n|\psi\rangle \\ &= \sum_n e^{itE_n(1-i\epsilon)}|n\rangle\langle n| \\ &\to_{\epsilon\to 0, t\to\infty} e^{itE_0(1-i\epsilon)}|0\rangle\langle 0|\psi\rangle \end{aligned} \tag{2.22}$$

[2]Someone interested in the exact expression of $[DpDq]$ would find diverging factor. However, this exact expression won't matter to us in what follows, so we may forget about this issue.

Then for any two fields ψ_1, ψ_2, for any two string of operators $\mathcal{O}_1, \mathcal{O}_2$, we have

$$\frac{\langle 0|\mathcal{O}_1|0\rangle}{\langle 0|\mathcal{O}_2|0\rangle} = \lim_{t_1,t_2\to\infty,\epsilon\to 0} \frac{\langle \psi_1|e^{-it_1H(1-i\epsilon)}\mathcal{O}_1 e^{-it_2H(1-i\epsilon)}|\psi_2\rangle}{\langle \psi_1|e^{-it_1H(1-i\epsilon)}\mathcal{O}_2 e^{-it_2H(1-i\epsilon)}|\psi_2\rangle} \tag{2.23}$$

Thus, for a sequence of fields $(\varphi_k)_k$ and a sequence of times $(t_k)_k$, recalling that $\varphi_k(t_k) = e^{it_kH}\varphi_k e^{-it_kH}$ with $\varphi_k = \varphi_k(0)$ (1.26), assuming that the $(\varphi_k)_k$ and $(t_k)_k$ are already time-ordered, we have

$$\begin{aligned}\langle \varphi_1(t_1)\dots\varphi_n(t_n)\rangle &= \frac{\langle 0|\varphi_1 e^{iH(t_2-t_1)}\varphi_2 e^{iH(t_3-t_2)}\dots e^{iH(t_n-t_{n-1})}\varphi_n|0\rangle}{\langle 0|e^{iH(t_n-t_1)}|0\rangle} \\ &= \frac{\langle \psi_1|e^{-iT_1H(1-i\epsilon)}\varphi_1 e^{iH(t_2-t_1)}\dots\varphi_n e^{-iT_2H(1-i\epsilon)}|\psi_2\rangle}{\langle \psi_1|e^{-i(T_1+T_2+t_1-t_n)H(1-i\epsilon)}|\psi_2\rangle}\end{aligned} \tag{2.24}$$

for $T_1, T_2 \to \infty, \epsilon \to 0$. By inserting sums of $|n\rangle\langle n|$ between each operator and by using extensively the path formalism, we obtain for the nominator

$$\lim_{T_1,T_2\to\infty,\epsilon\to 0} \int_{T_1}^{T_2} [\mathrm{d}\varphi]\psi_1^*(T_1)\psi_2(T_2)\varphi_1(t_1)\dots\varphi_n(t_n)e^{iS_\epsilon[\varphi(t)]} \tag{2.25}$$

Remembering that the fields ψ_1, ψ_2 where chosen arbitrarily, we can choose them such that $\psi_1(T_1) = \psi_2(T_2) = 1$. This way, we have

$$\langle \varphi_1(t_1)\dots\varphi_n(t_n)\rangle = \lim_{\epsilon\to 0} \frac{\int[\mathrm{d}\varphi]\varphi_1(t_1)\dots\varphi_n(t_n)e^{iS_\epsilon[\varphi(t)]}}{\int[\mathrm{d}\varphi]e^{iS_\epsilon[\varphi(t)]}} \tag{2.26}$$

With the change of coordinates $t \to -i\tau$, redefining $\varphi_k(-i\tau)$ as $\varphi_k(\tau)$, we finally have

$$\langle \varphi_1(\tau_1)\dots\varphi_n(\tau_n)\rangle = \frac{\int[\mathrm{d}\varphi]\varphi_1(\tau_1)\dots\varphi_n(\tau_n)e^{-S_E[\varphi(\tau)]}}{\int[\mathrm{d}\varphi]e^{-S_E[\varphi(\tau)]}} \tag{2.27}$$

with $S_E(\varphi(\tau)) = -iS(\varphi(t))$.

Remark. This last transformation leads to what is called the Euclidian formalism. To do the transformation, we have to assume that the correlation functions can be analytically continued from real time to imaginary time.

We call $Z = \int[\mathrm{d}\varphi]e^{-S_E[\varphi(\tau)]}$ the action functional, or the partition function in analogy with statistical mechanics. Defining it allows us to rewrite (2.27) as

$$\langle \varphi_1(x_1)\dots\varphi_n(x_n)\rangle = \frac{1}{Z}\int[\mathrm{d}\varphi]\varphi_1(x_1)\dots\varphi_n(x_n)e^{-S_E[\varphi]} \tag{2.28}$$

As expected, we find a relation close to Boltzmann's distribution, showing that the transitions requiring less energy are exponentially most probable to happen. We have also managed to change our landscape from operators to analysis, which is much more familiar to us and which will allow us to do extensive computations.

Finally, we should notice how the exact factors in the measure $[\mathrm{d}\varphi]$ cancel out, since everything is normalised in the final expression.

2.4 Symmetries and generators

Now that we have a good formulation of correlation functions, we would like to see how we can constrain them, to have easier computations. But how can we constrain them? What general constraints do we know on our theory? We know that in physics, in general, symmetries may severely constrain the dynamics of a system. Furthermore, due to the axioms of QFTs, we know that our theory is invariant under the Poincaré group. We should thus be able to use this symmetry at our advantage. In this subsection, we will see how this symmetry acts on our theory in the operator formalism. In the next, we will how it acts in the path-integral formalism. Finally, in a third subsection, we will see how by linking the two, we can get strong contraints on correlation functions called Ward identities.

Let's first see how a general symmetry manifests in a theory. We know that a symmetry is represented in a theory by a class of transformations, under which operators transform as (2.1). From the point of view of path-integrals, this manifests by the invariance of the action. Indeed, if the system acts the same way before and after a transformation, then the action should not change under this transformation.

Let S be the action of the system in the Euclidian formalism. Let $(\varphi_k)_k$ a sequence of fields, and $(x_k)_k$ a sequence of coordinates. Let $x_k \to x'_k$ a coordinate transformation under which the action is invariant (meaning $x_k \to x'_k$ represents a symmetry of the system). We write

$$\varphi'(x') = \mathcal{F}(\varphi(x)) \tag{2.29}$$

Usually, we would consider a class of such transformations, corresponding to the class of transformations representing a symmetry. We would then like to categorize all transformations of the class, or in other words find a basis for this class of transformations. We can in general parametrize the transformations by a set of infinitesimal parameters $(\omega_a)_a$ such that we have at first order

$$\begin{aligned} x'^{\mu} &= x^{\mu} + \omega_a \frac{\delta x^{\mu}}{\delta \omega_a} \\ \varphi'(x') &= \varphi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x) \end{aligned} \tag{2.30}$$

From this, we can directly define the basis for the transformations

Definition 2.4.1. We define the generator G_a of a transformation by the following expression

$$\varphi'(x) - \varphi(x) \equiv -i\omega_a G_a \varphi(x) \tag{2.31}$$

These generators are very convenient as they will allow us to easily consider the symmetries in computations.

Remark. Knowing that symmetries form in general a Lie group, the generators of a symmetry may be seen as the Lie algebra associated to the Lie group of the symmetry.

With (2.30), we can write

$$\varphi'(x') = \varphi(x') - \omega_a \frac{\delta x^{\mu}}{\delta \omega_a} \partial_{\mu} \varphi(x') + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x') \tag{2.32}$$

But according to (2.31),

$$iG_a\varphi(x') = \frac{1}{\omega_a}(\varphi(x') - \varphi'(x')) \tag{2.33}$$

Which results in the explicit form for the generator

$$iG_a\varphi = \frac{\delta x^\mu}{\delta\omega_a}\partial_\mu\varphi - \frac{\delta\mathcal{F}}{\delta\omega_a} \tag{2.34}$$

Now that we have seen the theory, let's switch to the practice. Let's compute the generators of the Poincaré group.

The Poincaré group is made up of translations and Lorentz transformations. Let's begin with translations. Let $x^\mu \to x^\mu + w^\mu = x'^\mu$ be an infinitesimal translation. Here, the index a is a spacetime index. We have

$$\frac{\delta x^\mu}{\delta\omega^\nu} = \delta^\mu_\nu \qquad \frac{\delta\mathcal{F}}{\delta\omega^\nu} = 0 \tag{2.35}$$

So the generators of translations are equal to

$$P_\nu = -i\partial_\nu \tag{2.36}$$

Now, let's turn to Lorentz transformations. Let $x^\mu \to \omega^\nu_\mu x^\mu = \omega_{\rho\mu}\eta^{\rho\nu}x^\mu = x'^\mu$ be an infinitesimal Lorentz transformation. ω must be antisymmetric, so we can write

$$\frac{\delta x^\mu}{\delta\omega_{\rho\nu}} = \frac{1}{2}(\eta^{\rho\mu}x^\mu - \eta^{\nu\mu}x^\rho) \tag{2.37}$$

It's effect on the field can be writen $\mathcal{F}(\psi) = L_\omega\psi$. At first order, we can write

$$L_\omega \simeq 1 - \frac{1}{2}i\omega_{\rho\nu}S^{\rho\nu} \tag{2.38}$$

with $S^{\rho\nu}$ an hermitian matrix. Using (2.34), we then get the formula for the generators of Lorentz transformations

$$L^{\rho\nu} = i(x^\rho\partial^\nu - x^\nu\partial^\rho) + S^{\rho\nu} \tag{2.39}$$

2.5 Symmetries and currents

Now that we have looked at symmetries in the operator formalism, let's try to see what can they tell us in the path integral formalism. As before, we should begin by studying an infinitesimal transformations, this time modifying the action of the system in the path integral formalism. Let $x^\mu \to x'^\mu = x^\mu + \omega_a\frac{\delta x^\mu}{\delta\omega_a}$ be an infinitesimal transformation, transforming a field as $\varphi(x) \to \varphi'(x') = \varphi(x) + \omega_a\frac{\delta\mathcal{F}}{\delta\omega_a}(x) = \mathcal{F}(\varphi(x))$. We have

$$\begin{aligned} S' &= \int d^dx\mathcal{L}(\varphi'(x), \partial_\mu\varphi'(x)) \\ &= \int d^dx'\mathcal{L}(\varphi'(x'), \partial'_\mu\varphi'(x')) \\ &= \int d^dx'\mathcal{L}(\mathcal{F}(\varphi(x)), \partial'_\mu\mathcal{F}(\varphi(x))) \\ &= \int d^dx|\frac{\partial x'}{\partial x}|\mathcal{L}(\mathcal{F}(\varphi(x)), \frac{\partial x^\mu}{\partial x'^\nu}\partial_\mu\mathcal{F}(\varphi(x))) \end{aligned} \tag{2.40}$$

But at first order, we have

$$\frac{\partial x'^{\nu}}{\partial x^{\mu}} \simeq \delta^{\nu}_{\mu} + \partial_{\mu}\left(\omega_a \frac{\delta x^{\nu}}{\delta \omega_a}\right) \qquad \frac{\partial x^{\nu}}{\partial x'^{\mu}} \simeq \delta^{\nu}_{\mu} - \partial_{\mu}\left(\omega_a \frac{\delta x^{\nu}}{\delta \omega_a}\right) \tag{2.41}$$

Moreover,

$$\det(1+E) \simeq 1 + \mathrm{Tr}(E) \tag{2.42}$$

So

$$\left|\frac{\partial x'}{\partial x}\right| \simeq 1 + \partial_{\mu}\left(\omega_a \frac{\delta x^{\mu}}{\delta \omega_a}\right) \tag{2.43}$$

Injecting these in (2.40), we get:

$$\begin{aligned} S' = \int d^d x \left(1 + \partial_{\mu}\left(\omega_a \frac{\delta x^{\mu}}{\delta \omega_a}\right)\right) \times \\ \mathcal{L}\left(\varphi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x), \left[\delta^{\nu}_{\mu} - \partial_{\mu}\left(\omega_a \frac{\delta x^{\nu}}{\delta \omega_a}\right)\right]\left(\partial_{\nu}\varphi(x) + \partial_{\nu}\left[\omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x)\right]\right)\right) \end{aligned} \tag{2.44}$$

We are interested in computing $\delta S = S' - S$. Expanding the Lagrangian to the first order, and defining the *Noether current* associated with the infinitesimal transformation as follows

$$j^{\mu}_a = \left[\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi)}\partial_{\nu}\varphi - \delta^{\mu}_{\nu}\mathcal{L}\right]\frac{\delta x^{\nu}}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi)}\frac{\delta \mathcal{F}}{\delta \omega_a} \tag{2.45}$$

we have

$$\delta S = -\int d^d x j^{\mu}_a \partial_{\mu}\omega_a \tag{2.46}$$

but, after integrating by parts,

$$\delta S = \int d^d x \partial_{\mu} j^{\mu}_a \omega_a \tag{2.47}$$

This is very interesting, as according to the laws of motion, δS should vanish for any choice of $\omega_a(x)$ due to Lagrange's principle. In particular, the Noether current is conserved with the motion

$$\partial_{\mu} j^{\mu}_a = 0 \tag{2.48}$$

In other words, this implies that for any continous symmetry in our theory, we can associated a conserved current.

Remark. This is called Noether's theorem, and serves as a fundamental fact in the modern study of symmetries. Some of Noether's currents are well-known in classical theories. For exemple, the current associated to translations in time is the energy, the current associated to translations in space is momentum, and the current associated to rotations is angular momentum.

To any conserved current j^{μ}_a, we can associate a conserved charge given by

$$Q_a = \int d^{d-1} x j^0_a \tag{2.49}$$

Now that we have understood how symmetries can be represented in the path-integral formalism, let's try to compute the currents associated to our symmetries. We should begin with translations. Recalling (2.35), we can compute the current assiociated with translations using the formula (2.45). Changing the index a to ν and moving it up using the metric tensor, we have

$$j^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial^\nu\varphi \tag{2.50}$$

We usually name this tensor the *canonical energy-momentum tensor*, or stress-energy tensor, and write it $T_c^{\mu\nu} \equiv j^{\mu\nu}$. The name of this tensor comes from the fact that the associated conserved charge obtained with (2.49) is the four-momentum, or energy-momentum

$$P^\nu = \int dx T_c^{0\nu} \tag{2.51}$$

We notice in particular that P^0 corresponds to the Hamiltonian. Intuitively, this is natural as the charge classically associated to translations in space is the momentum, and the charge classically associated to translations in time is the energy

This tensor may be modified, to make it more convenient to use. For example, adding the divergence of a tensor $B^{\mu\nu\rho}$ antisymmetric in the first two indices does not affect the conservation of the current. We can thus redefine our current at will by adding the divergence of such a tensor and still verify the associated ward identity. In particular, thanks to our theory having the Lorentz transformations as symmetry, we can find such a tensor such that the energy-momentum becomes symmetric. We won't go into the details of the computation, which are mostly technical. A demonstration can be found here [FMS96, p. 46]. We usually call the newly defined current the *Belinfante energy-momentum tensor*.

Let's now look at Lorentz transformations. Putting (2.37) and (2.38) into the formula for the current (2.45), we get the current associated with Lorentz transformations

$$j^{\mu\nu\rho} = T_c^{\mu\nu}x^\rho - T_c^{\mu\rho}x^\nu + i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}S^{\nu\rho}\varphi \tag{2.52}$$

But by rendering the energy-momentum tensor symmetric, we annihilated the last term, such that

$$j^{\mu\nu\rho} = T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu \tag{2.53}$$

We recognize here something that looks like angular momentum. However, this current is not reduced to angular momentum as it also contains the conserved current associated to boosts.

2.6 Ward identities

We have seen the effects of an infinitesimal transformation on the states of the theory and on its action. Let's see how it affects a correlation function. Let $X = \varphi_1(x_1)\dots\varphi_n(x_n)$ a string of fields. We write δ_ω its variation under the transformation parameterized by ω. Supposing that the transformation is a

symmetry of the theory, the correlation function of X is invariant under the transformation. We can thus rewrite the correlation function formula in the path integral formalism (2.27) after the transformation, and get

$$\langle X \rangle = \frac{1}{Z} \int [d\varphi'](X + \delta X) e^{-S[\varphi] - \int dx \partial_\mu j_a^\mu \omega_a(x)} \tag{2.54}$$

We may suppose that the measure is invariant under the transformation, meaning $[d\varphi'] = [d\varphi]$. Expanding the exponential at the first order, we can compute the variation of the correlation function

$$\langle \delta X \rangle = \int dx \partial_\mu \langle j_a^\mu(x) X \rangle \omega_a(x) \tag{2.55}$$

On the other hand, we can compute the variation of X explicitly using the definition of the transformation generators:

$$\begin{aligned} \delta X &= -i \sum_{k=1}^{n} \left(\varphi_1(x_1) \dots G_a \varphi_k(x_k) \dots \varphi_n(x_n)\right) \omega_a(x_k) \\ &= -i \int dx \omega_a(x) \sum_{k=1}^{n} \left(\varphi_1(x_1) \dots G_a \varphi_k(x_k) \dots \varphi_n(x_n)\right) \delta(x - x_k) \end{aligned} \tag{2.56}$$

But (2.55) and (2.56) are true for any infinitesimal transformation ω_a. We can thus go under the integral, and putting together the two ways to compute $\langle X \rangle$ we get

$$\partial_\mu \langle j_a^\mu(x) \varphi_1(x_1) \dots \varphi_n(x_n) \rangle = -i \sum_{k=1}^{n} \delta(x - x_k) \langle \varphi_1(x_1) \dots G_a \varphi_k(x_k) \dots \varphi_n(x_n) \rangle \tag{2.57}$$

This equation is called the Ward identity associated to the current j_a^μ. To any symmetry a theory has, one can associated its Ward identity by replacing the current and generator in the relation above.

Let's therefore do so for the symmetries of our theory. We start with translations, whose conserved current is the energy-momentum tensor. Injecting this tensor and the translation generator (2.36) in the general Ward identity (2.57), we immediatly get

$$\partial_\mu \langle T_\nu^\mu X \rangle = \sum_{k=1}^{n} \delta(x - x_k) \frac{\partial}{\partial x_k^\nu} \langle X \rangle \tag{2.58}$$

which is the Ward identity associated to translations.

We can do the same for Lorentz transformations. Putting their conserved current (2.53) along with the generator for Lorentz transformations (2.39) in the general Ward identity, we get

$$\partial_\mu \langle (T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu) X \rangle = \sum_{k=1}^{n} \delta(x - x_k) \left[(x_k^\nu \partial_k^\rho - x_k^\rho \partial_k^\nu) \langle X \rangle - i S_k^{\nu\rho} \langle X \rangle \right] \tag{2.59}$$

We can develop the divergence on the left hand side using Leibnitz rule. The derivative either acts on the energy-momentum tensor, and can be removed

using (2.58), or acts on x^μ and diseappears in a δ^ν_μ. We can thus reduce the expression above, to obtain

$$\langle (T^{\rho\nu} - T^{\nu\rho})X\rangle = -i\sum_{k=1}^{n}\delta(x-x_k)S_k^{\nu\rho}\langle X\rangle \tag{2.60}$$

This is the ward identity associated with Lorentz transformations.

2.7 The free boson, part 1

Now that we have accumulated some theory, we will end this section by looking at our first true exemple of quantum field theory. We will consider the simplest system possible, which is the free scalar field φ, a scalar field with action

$$\begin{aligned} S[\varphi] &= \int \mathrm{d}x\mathrm{d}t\mathcal{L}(\varphi,\dot\varphi,\nabla\varphi) \qquad \dot\varphi \equiv \frac{\partial\varphi}{\partial t} \\ \mathcal{L} &= \frac{1}{2}\left(\frac{1}{c^2}\dot\varphi^2 - (\nabla\varphi)^2 - m^2\varphi^2\right) \end{aligned} \tag{2.61}$$

Let's first decompose this Lagrangian. It contains 3 terms. The last one is the mass term $m^2\varphi^2$ quadratic with the field, as to control the amount of mass (number of particles) in the system. The middle one is the classical kinetic term $(\nabla\varphi)^2$ quadratic with the spatial derivative of the field. Finally, the first one is $\frac{1}{c^2}\dot\varphi^2$, a sort of momentum term quadratic in the time derivative of the field, with sign opposite to the other two so as to allow the formation of particles at a single level of energy.

For the rest of this section, we set $c = 1$. We consider this system in 2 dimensions, one temporal and one spatial dimension. This means that our theory is defined going from the Minkowski space in 2 dimensions $\mathcal{M}^2$ to the Hilbert space $\mathcal{M}^{2\mathbb{C}}$ of functions from the 2 dimensional Minkowski space to $\mathbb{C}$, that is the free Hilbert space generated from our base space. Intuitively, we are considerig particles moving on a line, whose probability of existence are given by the excitations of the field.

We can start to study the system by replacing the spatial dimension with a discrete set of points. We consider N points, with a lattice spacing of a. Moreover, we set a periodic boundary condition $\varphi_N = \varphi_0$. The Lagrangian (2.61) then becomes

$$\mathcal{L} = \sum_{k=0}^{N-1}\frac{a}{2}\left(\dot\varphi_k^2 - \frac{1}{a^2}(\varphi_{k+1}-\varphi_k)^2 - m^2\varphi_k^2\right) \tag{2.62}$$

From there, we define the canonical momentum conjugate to the variable φ_n:

$$\pi_n = a\dot\varphi_n \tag{2.63}$$

Rewriting the Lagrangian (2.62) in terms of the position and momentum, we get the Hamiltonian

$$H = \frac{1}{2}\sum_{k=0}^{N-1}\left(\frac{1}{a}\pi_k^2 - \frac{1}{a}(\varphi_{k+1}-\varphi_k)^2 - am^2\varphi_k^2\right) \tag{2.64}$$

We switch to the canonical quantization, by replacing the (φ_k) and (π_k) by operators, and by imposing at equal times canonical commutation relations[3]:

$$\begin{aligned}[\varphi_n, \pi_m] &= i\delta_{n,m} \\ [\pi_n, \pi_m] &= [\varphi_n, \varphi_m] = 0\end{aligned} \tag{2.65}$$

The system is invariant under translations, and so is the Hamiltonian. This motivates the use of Fourier transforms.

$$\begin{aligned}\tilde{\varphi}_k &= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{-2\pi i k j/N} \varphi_j \\ \tilde{\pi}_k &= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{-2\pi i k j/N} \pi_j\end{aligned} \tag{2.66}$$

φ_k and π_k are real so $\tilde{\varphi}_k^\dagger = \tilde{\varphi}_{-k}$ and $\tilde{\pi}_k^\dagger = \tilde{\pi}_{-k}$. We have $[\tilde{\varphi}_n, \tilde{\pi}_m^\dagger] = i\delta_{n,m}$

With these, the Hamiltonian (2.64) becomes

$$H = \frac{1}{2} \sum_{k=0}^{N-1} \left(\frac{1}{a} \tilde{\pi}_k \tilde{\pi}_k^\dagger + a \tilde{\varphi}_k \tilde{\varphi}_k^\dagger \left[m^2 + \frac{2}{a^2} \left(1 - cos(\frac{2\pi k}{N}) \right) \right] \right) \tag{2.67}$$

This is exactly the Hamiltonian for a system of uncoupled harmonic oscillators, with frequencies

$$\omega_k = \sqrt{m^2 + \frac{2}{a^2} \left(1 - cos(\frac{2\pi k}{N}) \right)} \tag{2.68}$$

We can then define the annihilation and creation operators

$$\begin{aligned}a_k &= \frac{1}{\sqrt{2a\omega_k}} (a\omega_k \tilde{\varphi}_k + i\tilde{\pi}_k) \\ a_k^\dagger &= \frac{1}{\sqrt{2a\omega_k}} (a\omega_k \tilde{\varphi}_k^\dagger - i\tilde{\pi}_k^\dagger)\end{aligned} \tag{2.69}$$

We do have

$$[a_n, a_m^\dagger] = \delta_{n,m} \tag{2.70}$$

The Hamiltonian rewrites nicely as

$$H = \sum_{k=0}^{N-1} (a_k^\dagger a_k + \frac{1}{2}) \omega_k \tag{2.71}$$

We see there that a_k and $a_k^\dagger$ indeed act as annihilation and creation operators, with $a_k^\dagger a_k$ the density operator. We define the vacuum $|0\rangle$ such that for all k, $a_k|0\rangle = 0$. From this, any state can be computed by repeatedly applying creation operators on the vacuum.

We can let $a \to 0$ and $N \to \infty$ simultaneously, such that Na stays constant. This way, we get back to the continuous system in 2 dimensions. φ_k becomes

[3] These commutation relations are induced by Schrödinger's equation, and are deemed as common knowledge in quantum mechanics and QFTs

$\varphi(x)$, $\frac{1}{a}\pi_n$ becomes $\pi(x) = \dot{\varphi}(x)$. $a\sum_{k=0}^{N-1}$ becomes $\int \mathrm{d}x$, and $\delta_{n,m}$ becomes $a\delta(x-x')$.

Similarly, the Fourier indices k are replaced by the momentum $p = \frac{2\pi k}{V}$. $\sum_{k=0}^{N-1}$ becomes $\frac{V}{2\pi}\int \mathrm{d}p$, and a_k is replaced by $\frac{1}{V}a(p)$, with energy $\omega(p) = \sqrt{m^2+p^2}$.

The simplest states are of the form $a^\dagger(p)|0\rangle$, with energy $\omega(p) = \sqrt{m^2+p^2}$. As this is the well-known dispersion relation typical of relativistic particles, we can interpret these states as elementary particles. As expected, we are indeed dealing with a system of particles where the particles correspond to excitations of the field. Moreover, general states aren't affected by switching two particles

$$a^\dagger(p)a^\dagger(p')|0\rangle = a^\dagger(p')a^\dagger(p)|0\rangle \tag{2.72}$$

We are thus considering bosons. As there are no interactions between these bosons, we call this theory the free boson theory.

Now that we have understood what we are dealing with, we would like to compute the propagator of the free boson field, or in other words its 2-points correlation function. We define

$$K(x,y) \equiv \langle\varphi(x)\varphi(y)\rangle \tag{2.73}$$

For the sake of simplicity and for reasons that will come later on, we suppose that $m = 0$. We can rewrite (2.61) as

$$S = \frac{1}{2}\int d^2x d^2y \varphi(x)A(x,y)\varphi(y) \qquad A(x,y) = -\partial^2\delta(x,y) \tag{2.74}$$

But (2.27) gives an expression of the correlation function according to the action. In fact, this expression can be reduced to $K(x,y) = A^{-1}(x,y)$, or equivalently

$$-\partial_x^2 K(x,y) = \delta(x,y) \tag{2.75}$$

This is a consequence of the properties of Gaussian integrals, which we will not go through but are discussed in [FMS96, p. 51].

Because of translation and rotation invariance, $K(x,y)$ should only depend on $|x-y|$. We thus write $K(x,y) = K(r)$. Integrating (2.75) over a disk centered around y then results in

$$1 = -2\pi r K'(r) \tag{2.76}$$

whose solution is

$$K(r) = -\frac{1}{2\pi}\ln(r) + C \tag{2.77}$$

with C a constant. Returning to correlation functions, we have

$$\langle\varphi(x)\varphi(y)\rangle = -\frac{1}{4\pi}\ln(x-y)^2 + C \tag{2.78}$$

Note how we didn't have to use any of the constraints we derived earlier to compute this correlation function. However, we still had to use some sort of constraints, as we used the symmetries of the system to constrain the form of K.

The energy-momentum tensor associated to this system is

$$T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} \eta_{\mu\nu} \partial_\rho \varphi \partial^\rho \varphi \tag{2.79}$$

However, we won't study it nor apply the Ward identity to it now. We will get better tools in the next section, which will allow us to derive the correlation function of the energy-momentum tensor in a much more efficient way.

Chapter 3

2D CFTs

We should now introduce and study the basic properties of the main type of theory of this paper, which is 2 dimensional conformal field theories (2D CFTs).

3.1 Conformal field theories

First of all, we need to introduce what is a conformal field theory (CFT). To understand the motivation for it, let's first recall how we study QFTs. One of the axioms (2.1) of QFT states that a QFT should always be invariant under the Poincaré group. From this strong condition, we get a representation of the symmetry on the theory, and are able to derive strong constraints such as the Ward identities seen in subsection 2.6. But in most cases, these constraints are not strong enough to help us completely determine the form of correlation functions. In fact, most of the theories one could define with a Lagrangian are difficult to study, if not impossible.

The idea of CFTs is to add a symmetry to the theory, to upgrade the Poincaré invariance to a bigger one, such that this condition is strong enough to allow us to compute correlation functions. This bigger group is the group of conformal transformation, hence the name conformal field theory.

To formally define CFTs, we first need to define what is a conformal transformation.

Definition 3.1.1. A coordinate transformation is called *conformal* if it preserves angle, or equivalently if it is a local rescaling of the metric. Mathematically speaking, g is a conformal transformation if under a coordinate transformation $x^\mu \to w^\mu(x)$ the spacetime metric transforms as

$$g'_{\mu\nu}(w) = \Lambda(x) g_{\mu\nu}(x) \tag{3.1}$$

Definition 3.1.2. A conformal transformation is said to be *global* if it is invertible.

We call a conformal field theory (or a CFT) any quantum field theory invariant through global conformal transformations. In term of axioms, it is equivalent to extending the unitary representation of the Poincaré group to the global

conformal group, and to ensure the fields are covariant under global conformal transformations. A 2D CFT is then a CFT whose base spacetime is (at least locally) 2-dimensional.

The group of global conformal transformations mostly includes the Poincaré group and dilatations ($x \to \lambda x$ for $\lambda \in \mathbb{R}$). Intuitively we can therefore expect QFTs to be CFTs if they are invariant under dilatations (zooming or distancing), or equivalently we can expect general systems to be invariant under conformal transformations if they are invariant under translations, rotations, boosts, and dilatations. However, this should remain a simple intuition as the conformal group also includes other less intuitive transformations. In fact, it is precisely generated by translations, boosts, dilatations, and special conformal transformations, given for $b^\mu \in \mathcal{M}$ by

$$x^\mu \to \frac{||x^\beta||^2}{||x^\beta - b^\beta x^\gamma x_\gamma||^2}(x^\mu - b^\mu x^\gamma x_\gamma) \tag{3.2}$$

Note how due to the invariance through dilatations, in a CFT, the notion of distance diseappear. In particular, there is no notion of far and near. In fact, the space underlying CFTs is often assumed to be compact, or is made compact. We will return to this point in more details later on. Similarly, the notion of mass must also diseappear.

Do CFTs exist in real life? Truth is, rarely. We have two main exemples of CFTs that may be used to model our world. The first one is string theory, which is a kind of CFT used to try to explain the standard model and to try to make a quantum theory of gravity. The second exemple come from condensed matter physics, and corresponds to condensed matter systems near phase transitions. This kind of system is usually invariant under the conformal group because at a mesoscopic scale, a system in transition with equal purcentage of each phases is invariant under dilatations.

3.2 2D conformal transformations

Conformal transformations are especially interesting in 2 dimensions. The idea behind CFTs is to enforce a large number of symmetries to put strong constraints on our theory. And in 2 dimensions, the space of conformal transformations is infinite dimensional, allowing for very, very strong constraints. For this reason, we will from now on focus on 2D CFTs.

Let's prove what we just said above, and compute the space of conformal transformations in 2 dimensions. We take a 2 dimensional space with the Euclidian metric. With Einstein's notations, for a coordinate transformation $x^\mu \to w^\mu(x)$, the metric tensors transforms as

$$g'^{\mu\nu} = \left(\frac{\partial w^\mu}{\partial x^\alpha}\right)\left(\frac{\partial w^\nu}{\partial x^\beta}\right) g^{\alpha\beta} \tag{3.3}$$

For it to be a conformal transformation, for all $\mu, \nu \in \{0, 1\}$, we must have

$$g'^{\mu\nu} = \Lambda g^{\mu\nu} \tag{3.4}$$

With Λ depending on the position. g is the Euclidian metric, so we have $g^{01} = g^{10} = 0$ and $g^{11} = g^{00} = 1$. The condition for the transformation to be conformal thus becomes

$$g'^{00} = g'^{11} = 0 \tag{3.5}$$

$$g'^{01} = g'^{10} \tag{3.6}$$

Adding (3.3) to the above equations, we get the following two conditions:

$$\frac{\partial w^0}{\partial x^0}\frac{\partial w^1}{\partial x^0} + \frac{\partial w^0}{\partial x^1}\frac{\partial w^1}{\partial x^1} = 0 \tag{3.7}$$

$$\left(\frac{\partial w^0}{\partial x^0}\right)^2 + \left(\frac{\partial w^0}{\partial x^1}\right)^2 = \left(\frac{\partial w^1}{\partial x^0}\right)^2 + \left(\frac{\partial w^1}{\partial x^1}\right)^2 \tag{3.8}$$

Writing $\partial_0 \equiv \frac{\partial}{\partial x^0}$ and $\partial_1 \equiv \frac{\partial}{\partial x^1}$, the above conditions can be resumed to

$$\partial_0 w_1 = \pm\partial_1 w_0 \qquad \partial_0 w_0 = \mp\partial_1 w_1 \tag{3.9}$$

These exactly corresponds to the holomorphic and anti-holomorphic Cauchy-Riemann equations. Therefore, w is either an holomorphic or antiholomorphic function.

To see things better, we use the Wick rotation on the plane. We define

$$\begin{aligned} z &\equiv x^0 + \mathrm{i}x^1 \quad \partial \equiv \frac{1}{2}(\partial_0 - \mathrm{i}\partial_1) \\ \bar{z} &\equiv x^0 - \mathrm{i}x^1 \quad \bar{\partial} \equiv \frac{1}{2}(\partial_0 + \mathrm{i}\partial_1) \end{aligned} \tag{3.10}$$

Note that the metric tensor thus changes to

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \tag{3.11}$$

With these coordinates, the holomorphic Cauchy-Riemman equation become

$$\bar{\partial} w(z, \bar{z}) = 0 \tag{3.12}$$

whose solutions are all (analytical) function $z \to w(z)$. On the other hand, the antiholomorphic Cauchy-Riemman equation becomes

$$\partial w(z, z) = 0 \tag{3.13}$$

whose solutions are all (analytical) function $\bar{z} \to w(\bar{z})$.

Therefore, we see that both equations decouple and have solutions along their own dimension. The holomorphic equation gives functions along the dimension generated by z, whilst the antiholomorphic equation gives functions along the dimension generated by $\bar{z}$. For this reason, we call the dimension generated by z the holomorphic dimension, and the other one the antiholomorphic dimension. This decoupling is a deep fact about the nature of 2D CFTs, which holds true in any 2D CFT. Its consequences will completely split the theory in 2 parts, as we will see in 3.9. For this reason, we will sometimes call one part of it a "chiral" part, and by name abuse we will sometimes call the holomorphic dimension the

chiral dimension, and the antiholomorphic dimension the antichiral dimension. The reason for this decoupling can be understood intuitively, but we will see the reason later on when studying its effects on the space of states. To preserve this decoupling, we will in the following almost always consider these coordinates instead of the more natural ones.

We have seen the form of general conformal transformations. However, a CFT is not invariant under any kind of conformal transformation: it is only under global conformal transformation. Indeed, the transformation should at least be invertible for it to be a symmetry of the system. In fact, every symmetry should form a group. Let's see what is the group underlying global conformal transformations.

Let f be a global conformal transformation on a two-dimensional space. f is also a local conformal transformation, so according to our previous computations it should either depend on z or $\bar{z}$. We can suppose without loss of generality that it depends on z. We can also assume f is analytical. Moreover, f must be invertible so injective: f can't have essential singularities nor branch points. Thus, there exist $P, Q \in \mathbb{C}[X]$ such that

$$f(z) = \frac{P(z)}{Q(z)} \tag{3.14}$$

If P has multiple roots, f is not injective. The same goes if Q has multiple roots. As such, there exist $a, b, c, d \in \mathbb{C}$ such that

$$f(z) = \frac{az + b}{cz + d} \tag{3.15}$$

For f to be invertible, the determinant $ad - bc$ must be different than 0. As the choice of a, b, c, d is not unique in (3.15), we can normalize and choose $ad - bc = 1$. Reciprocally, we can easily verify that for any $a, b, c, d, e, f, g, h \in \mathbb{C}$ such that $ad - bc = 1, eh - fg = 1$, the transformation $z, \bar{z} \to \frac{az+b}{cz+d}, \frac{e\bar{z}+f}{g\bar{z}+h}$ is a global conformal transformation.

Therefore, the group of 2D global conformal transformation depending on z is isomorphic to $SL(2, \mathbb{C})$. As the exact same things happens to transformations depending on $\bar{z}$, we conclude that the complete conformal group is isomorphic to twice that. To remember one depends on the holomorphic dimension and one depends on the antiholomorphic one, we may write the conformal group $SL(2, \mathbb{C}) \times \overline{SL(2, \mathbb{C})}$.

3.3 Primary and quasi-primary fields

With this decoupling understood, we will now preemptively give some vocabulary, which will prove useful in the study of conformal transformations and CFTs. Moreover, we will also now derive a strong constraint from this conformal invariance. But first, some vocabulary!

Let $\varphi(z, \bar{z})$ be a field in a 2D CFT.

Definition 3.3.1. We say φ is *holomorphic* (or *chiral*) if it only depends on z, and we say it is *anti-holomorphic* (or *anti-chiral*) if it only depends on $\bar{z}$

A CFT is invariant through the conformal group, so through rescaling too. In particular, under the rescalings $z \to \lambda z$ and $\bar{z} \to \bar{\lambda}\bar{z}$, a field changes as

$$\varphi(z, \bar{z}) \to \lambda^h \bar{\lambda}^{\bar{h}} \varphi(\lambda z, \bar{\lambda}\bar{z}) \tag{3.16}$$

We call h its holomorphic (scaling) dimension, and $\bar{h}$ its anti-holomorphic (scaling) dimension. We also call the couple $(h, \bar{h})$ the conformal dimensions of φ, $\Delta = h + \bar{h}$ its scaling dimension, and $s = h - \bar{h}$ its spin. Note that Δ and s are given these names as they describe the behaviour of the field under dilatations and rotations, respectively.

Definition 3.3.2. A field φ is said to be *quasi-primary* if for any global conformal transformation $z \to f(z)$, the field transforms as

$$\varphi(z, \bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial f}{\partial \bar{z}}\right)^{\bar{h}} \varphi(f(z, \bar{z})) \tag{3.17}$$

Definition 3.3.3. A field φ is said to be *primary* if for any, possibly local, conformal transformation $z \to f(z)$, the field transforms as described by equation (3.17).

Remark. We have considered transformations of the form $z \to f(z)$ in the definition of quasi-primary and primary fields. Of course, the same should hold true for transformations of the form $\bar{z} \to f(\bar{z})$. We have simply considered transformations depending on the holomorphic dimension to lighten the notations, without loss of generality.

Quasi-primary and primary fields will prove useful when constructing the states of a general CFT, in subsection 3.9.

Now that we have aquired some vocabulary, we can use the conformal invariance to determine a general constraint on correlation functions, as well as the exact form of the 2-points correlation function of primary fields. We may also do the same for higher order correlation functions, but the computations become very difficult and the results are not so satisfactory nor intuitive. We won't show it in this paper for these reasons, though it can be found in [FMS96].

We consider the system of a 2D CFT. Let S be the action of the system in the Euclidian formalism. Let $(\varphi_k)_k$ a sequence of fields, and $(x_k)_k$ a sequence of coordinates. Let $x_k \to x'_k$ a coordinate transformation under which the action is invariant. We write as in 2.4

$$\varphi'(x') = \mathcal{F}(\varphi(x)) \tag{3.18}$$

Assuming as in 2.4 that the measure is invariant through the transformation, we have

$$\begin{aligned}
\langle \varphi_1(x'_1) \dots \varphi_n(x'_n) \rangle &= \frac{1}{Z} \int d[\varphi] \varphi_1(x'_1) \dots \varphi_n(x'_n) e^{-S[\varphi]} \\
&= \frac{1}{Z} \int d[\varphi'] \varphi'_1(x'_1) \dots \varphi'_n(x'_n) e^{-S[\varphi']} \\
&= \frac{1}{Z} \int d[\varphi] \mathcal{F}(\varphi_1(x_1)) \dots \mathcal{F}(\varphi_n(x_n)) e^{-S[\varphi]} \\
&= \langle \mathcal{F}(\varphi_1(x_1)) \dots \mathcal{F}(\varphi_n(x_n)) \rangle
\end{aligned} \tag{3.19}$$

In particular, recalling (3.16), writing the scaling dimensions of the fields $(\Delta_k)_k$, for any $\lambda \in \mathbb{R}$, we have

$$\langle \varphi_1(\lambda x_1) \dots \varphi_n(\lambda x_n) \rangle = \lambda^{-\Delta_1 \cdots - \Delta_n} \langle \varphi_1(x_1) \dots \varphi_n(x_n) \rangle \tag{3.20}$$

This is already a strong constraint imposed by the invariance under dilatations, actually valid in CFTs of any dimension. But we can go even further if we specialize this to 2 dimensions, and add the contribution of the other symmetries.

First, let $(\varphi_k)_k$ be a sequence of primary fields, with conformal dimensions $(h_k)_k$ and $(\bar{h}_k)_k$. According to (3.17), for a conformal transformation of the form $z \to w, \bar{z} \to \bar{w}$,(3.19) becomes

$$\begin{aligned} &\langle \varphi_1(w_1, \bar{w}_1) \dots \varphi_n(w_n, \bar{w}_n) \rangle \\ &\qquad = \prod_{k=1}^{n} \left(\frac{\mathrm{d}w}{\mathrm{d}z} \right)_{w=w_k}^{-h_k} \left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}} \right)_{\bar{w}=\bar{w}_k}^{-\bar{h}_k} \langle \varphi_1(z_1, \bar{z}_1) \dots \varphi_n(z_n, \bar{z}_n) \rangle \end{aligned} \tag{3.21}$$

This general relation on the correlation function of primary fields can prove useful. Now look at the 2-points correlation function of primary fields in a 2D CFT. Due to rotation and translation invariance, we have in any dimension that

$$\langle \varphi_1(x_1) \varphi_2(x_2) \rangle = f(|x_1 - x_2|) \tag{3.22}$$

Specializing this to 2 dimensions, $|x_1, x_2|$ becomes $((z_1 - z_2)(\bar{z}_1 - \bar{z}_2))^{\frac{1}{2}}$ with the coordinates defined by (3.10). The above equation can therefore be rewritten

$$\langle \varphi_1(z_1, \bar{z}_1) \varphi_2(z_2, \bar{z}_2) \rangle = f((z_1 - z_2)(\bar{z}_1 - \bar{z}_2)) \tag{3.23}$$

But with (3.20), this is constrained to

$$\langle \varphi_1(z_1, \bar{z}_1) \varphi_2(z_2, \bar{z}_2) \rangle = \frac{C}{(z_1 - z_2)^{h_1 + h_2} (\bar{z}_1 - \bar{z}_2)^{\bar{h}_1 + \bar{h}_2}} \tag{3.24}$$

with C a constant depending on the 2 fields. Finally, using the covariance of the correlation function with special conformal transformations, we get an equation implying that the conformal dimensions of the two fields must be equal, as shown in [FMS96, p. 105]. If the conformal dimensions of the two fields are different, the correlation function necessarily vanishes. Else, we have

$$\langle \varphi_1(z_1, \bar{z}_1) \varphi_2(z_2, \bar{z}_2) \rangle = \frac{C}{(z_1 - z_2)^{2h} (\bar{z}_1 - \bar{z}_2)^{2\bar{h}}} \tag{3.25}$$

with $h = h_1 = h_2$, $\bar{h} = \bar{h_1} = \bar{h_2}$. This super strong relation brings down the computation of any 2-points correlation function of primary fields to the determination of a single constant, and only using conformal invariance. We now see the power of this symmetry in action.

3.4 Radial quantization

We have now seen some elementary effects of the conformal symmetry, though we haven't started a proper methodical study of its effects on the symmetry.

This study will start next subsection but before, we would like to set up correctly the space we are working on.

Let's consider the space of our theory. It is a 2 dimensional Euclidian space, where one dimension corresponds to time whilst the other corresponds to space. We have canonical coordinates on this space, along two orthogonal axis, usually denoted by (x, y). One of the coordinates usually corresponds to the time dimension, whilst the other usually corresponds to the space dimension. Using these, we have defined our complex coordinates with (3.10), which will prove useful to distinguish the 2 chiral parts of our theory. In these coordinates, we can classically express time as $\frac{1}{2}(z + \bar{z})$, and space as $\frac{i}{2}(\bar{z} - z)$. However, the space and time are arbitrarily chosen. One could in fact choose any basis of the 2 dimensional space, and attribute time to one axis and space to the other axis. The only condition is that the basis must be orthogonal.

Moreover, as we are considering a conformal field theory, one can map 0 to ∞ and ∞ to 0 at will using the conformal symmetries. We can therefore assume without too much danger that compactifying the space of the theory won't affect the theory much. Compactifying the space dimension, one could consider a cylinder $\mathbb{S}^1 \times \mathbb{R}$ of circumference L where $\mathbb{S}^1$ corresponds to the space dimension, and $\mathbb{R}$ corresponds to the time dimension. In this space, we have the canonical coordinates (x, t) where we identified (x, t) and $(x + L, t)$ for any x, t. Furthermore, adding a point at $-\infty$ in the time dimension, we can map this cylinder back to the complex plane through the map

$$(x, t) \to e^{\frac{2\pi}{L}(t+ix)} \tag{3.26}$$

This way, we have defined a new basis for our Euclidian space, and new space and time dimensions. This choice of basis for time and space is called the radial quantization.

We want to define an Hermitian conjugation on this space, taking into account the newly defined time and space dimensions. To do so, let's consider an interacting 2D CFT, in which there is a field φ. Just like we saw in 2.2, we would like to compute the interactions happening in the theory by considering a state φ_{in} "entering" the space at a time $-\infty$, and by computing its probability $\mathbb{P} = \langle \varphi_{\text{out}} | \varphi_{\text{in}} \rangle$ to become a state φ_{out} "exiting" the system at a time $+\infty$.

Remark. This should stay as a motivation for correlation functions in general, as the scattering method doesn't work in CFTs. The basic assumption of the scattering method is to say that states are asymptotically free. However, in a CFT, due to the invariance under dilatations, states in an interacting field theory are never free.

In the radial quantization, we can easily compute φ_{in}. We want

$$\varphi_{\text{in}} \propto \lim_{t \to -\infty} \varphi(x, t) \tag{3.27}$$

But in the complex plane, 0 corresponds to $t = -\infty$. We have

$$|\varphi_{\text{in}}\rangle = \lim_{z, \bar{z} \to 0} \varphi(z, \bar{z})|0\rangle \tag{3.28}$$

And

$$\langle\varphi_{\text{out}}| = |\varphi_{\text{in}}\rangle^\dagger \tag{3.29}$$

In the Euclidian formalism we have been using up until now, we considered a complex time $\tau = it$, such that under Hermitian conjugation time would transform as $\tau \to -\tau$, whilst space would be left unchanged. We would like to keep this kind of transformation. With radial quantization, reversing time while leaving space unchanged corresponds to the map $z \to \frac{1}{z*}$. One could be tempted to define Hermitian conjugation as follows

$$\varphi(z,\bar{z})^\dagger = \varphi(\frac{1}{\bar{z}},\frac{1}{z}) \tag{3.30}$$

But if we did so, we would have

$$\begin{aligned}\langle\varphi_{\text{out}}|\varphi_{\text{in}}\rangle &= \lim_{z,\bar{z},w,\bar{w}\to 0}\langle 0|\varphi(z,\bar{z})^\dagger\varphi(w,\bar{w})|0\rangle \\ &= \lim_{z,\bar{z},w,\bar{w}\to 0}\langle 0|\varphi(\frac{1}{\bar{z}},\frac{1}{z})\varphi(w,\bar{w})|0\rangle \\ &= \lim_{z,\bar{z}\to+\infty}\langle 0|\varphi(\bar{z},z)\varphi(0,0)|0\rangle\end{aligned} \tag{3.31}$$

Supposing that φ is a primary field, recalling (3.25), we have

$$\langle 0|\varphi(\bar{z},z)\varphi(0,0)|0\rangle \propto z^{-2\bar{h}}\bar{z}^{-2h} \tag{3.32}$$

$\langle\varphi_{\text{out}}|\varphi_{\text{in}}\rangle$ would thus always be equal to 0, which is not what we want. In order to make sense of $\langle\varphi_{\text{out}}|\varphi_{\text{in}}\rangle$, we must define the Hermitian conjugate of a field as

$$\varphi(z,\bar{z})^\dagger = \bar{z}^{-2h}z^{-2\bar{h}}\varphi(\frac{1}{\bar{z}},\frac{1}{z}) \tag{3.33}$$

The radial quantization allows us to do many things, one of them being the operator-state correspondance. Let us consider a field φ. Up until now, we have sometimes associated a state to the field by saying

$$\varphi(t) = e^{iTH}\varphi e^{-iTH}, \quad |\varphi\rangle = \varphi|0\rangle \tag{3.34}$$

However, this attribution of fields to states is very arbitrary and can only be kept locally in a demonstration, as it obviously depends on an arbitrary point x and time frame t_0 such that $\varphi(t) = \varphi(T + t_0, x)$. In particular we do not always know how to translate a state through space, and we must therefore consider fields at the same point in space when using this correspondance. But these limitations change when in the radial quantization. Indeed, in the radial quantization, time is compactified at $-\infty$ and at this point, all of the space dimension merge into one point. Therefore, we may consider the fields taken at $(z,\bar{z}) = (0,0)$, which corresponds to the far past for all of space. We set

$$|\varphi\rangle = \lim_{z\to 0,\bar{z}\to 0}\varphi(z,\bar{z})|0\rangle \tag{3.35}$$

which gives a correspondance between fields and states. We can easily see that this correspondance is injective (by rigidity of analytical functions) and surjective by the axioms of QFT. Therefore, it gives a bijection between local operators at one point and states, or in other words between fields and states.This bijection is called the state-operator correspondance. This fact is fundamental in CFTs, and greatly simplifies the study of the space of states $\mathcal{H}$.

In what will follow, we will speak of states or operator equivalently. In particular, the space of operators should be understood as $\mathcal{H}$.

3.5 Conformal generators

Now that the space we are working on is properly set, let's investigate properly the effects of conformal symmetry on our theory. We would eventually like to derive constraints on the correlation functions as we did in 2.4, 2.5 and 2.6. The first step is thus to do as in 2.4, and to compute the generators of the symmetry.

Remark. Note that at the level of infinitesimal transformations (as we consider when computing the generators), any transformation is invertible. We will therefore have generators generating all conformal transformations present in our theory, instead of just the generators of $SL(2,\mathbb{C}) \times \overline{SL(2,\mathbb{C})}$.

We consider an infinitesimal conformal transformation ϵ. As we have shown, conformal transformations either depend on the holomorphic or the antiholomorphic dimension. Without loss of generality, we will consider one depending on the holomorphic dimension, $\epsilon = \epsilon(z)$. We may assume as previously that ϵ can be expanded in a serie. However, since it may not be invertible, we should expand it in its Laurent serie

$$z' = z + \epsilon(z) = z - \sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1} \tag{3.36}$$

We have a natural basis for such transformations, indexed by n. Let's try to find the generator of the transformation made by the n^{th} term. Applying the transformation on a dimensionless field $\phi(z)$ gives $\phi'(z) = \phi(z) + \delta\phi(z) = \phi(z) + \epsilon(z)\partial\phi(z)$. Thus, the generator associated to the n^{th} term of the serie is

$$l_n = -z^{n+1}\partial \tag{3.37}$$

The family $(l_n)_{n\in\mathbb{Z}}$ is the family of generators of conformal transformations depending on the holomorphic dimension. We also have the same family in the antiholomorphic dimension:

$$\bar{l}_n = -z^{n+1}\bar{\partial} \tag{3.38}$$

We explicitely see how the space of conformal transformation in 2D is infinite dimensional, hinting at the strength of the constraints it will provide. If we try to compute the conformal algebra, we have

$$\begin{aligned}[l_m, l_n] &= -z^{m+1}\partial(-z^n + 1)\partial + z^{n+1}\partial(-z^m + 1)\partial \\ &= (n-m)z^{m+1+n}\partial \\ &= (m-n)l_{m+n}\end{aligned} \tag{3.39}$$

These bracket relations define what is called a *Witt algebra*. We also have as expected

$$[\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n} \tag{3.40}$$

$$[l_m, \bar{l}_n] = 0 \tag{3.41}$$

We could expect to have this algebra present in the space of operators of any CFT, as a CFT is invariant under the conformal group. However, the algebra every CFT has is a slightly modified version of it, modified by a central extension, called the Virasoro algebra. We will proceed to proving this and get an explanation of the reason why in the next 2 subsections.

3.6 Conformal Ward identities

Now that we have derived the general generators of conformal transformations, we may specialize these to get the generator and current associated to dilatation, from which we should be able to derive a Ward identity.

First, what precisely is the generator of dilatations? Let $x^\mu \to (1+\omega)x^\mu$ be a dilatation. According to 3.16, $\mathcal{F}(\phi) = (1+\omega)^{-\Delta}\phi \simeq (1-\omega\Delta)\phi$ with Δ the scaling dimension of the field. We thus have

$$\frac{\delta x^\mu}{\delta \omega^\nu} = x^\mu \delta^\mu_\nu \qquad \frac{\delta \mathcal{F}}{\delta \omega^\nu} = -\Delta \tag{3.42}$$

So the generator of dilatations is equal to

$$D = -ix^\nu \partial_\nu - i\Delta \tag{3.43}$$

Now, what is its current? Injecting (3.42) into the formula for the current (2.45), we have

$$\begin{aligned} j^\mu &= -\mathcal{L}x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} x^\nu \partial_\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \\ &= T^\mu_\nu x^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \end{aligned} \tag{3.44}$$

This result can be improved. In particular, it can be shown that one may modify (once again) the energy-momentum tensor to make it traceless, thanks to the conformal symmetry of the system. We won't prove it here as the proof is rather technical, but a complete demonstration in 2 dimensions can be found in [FMS96, p. 107]. Using this and the fact that by definition of the current, $\partial_\mu j^\mu = 0$, we have

$$\partial_\mu T^\mu_\nu x^\nu + \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi = 0 \tag{3.45}$$

which implies

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi = 0 \tag{3.46}$$

Therefore, we have

$$j^\mu = T^\mu_\nu x^\nu \tag{3.47}$$

With these tools, we may compute the Ward identity associated to dilatations. Inserting (3.43) and (3.47) in the general Ward identity (2.57), we get

$$\partial_\mu \langle T^\mu_\nu x^\nu X \rangle = -\sum_{k=i}^{n} \delta(x - x_k) \left[x^\nu_k \frac{\partial}{\partial x^\nu_k} \langle X \rangle + \Delta_k \langle X \rangle \right] \tag{3.48}$$

Once again, we can simplify using Leibnitz rule

$$\langle T^\mu_\mu X \rangle = -\sum_{k=i}^{n} \delta(x - x_k) \Delta_k \langle X \rangle \tag{3.49}$$

which results in the ward identity associated with dilatations.

Up until now, we have only derived the Ward identities for a system in any dimension. But we know that the conformal symmetry is especially strong in 2 dimensions, and we have seen that a lot of things look better in 2 dimensions in the coordinates given by (3.10). Let's therefore rewrite the Ward identities we know of in these coordinates.

To do so, we need to have a few computational tools in these coordinates. First, we should recall that the metric is given by (3.11). Moreover, in these coordinates, the antisymmetric tensor takes the form

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2}i \\ -\frac{1}{2}i & 0 \end{pmatrix} \tag{3.50}$$

We will also use the identity

$$\delta(x) = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z} = \frac{1}{\pi}\partial_z\frac{1}{\bar{z}} \tag{3.51}$$

We will not prove this indentity, which can be derived from contour integrals. A detailed justification is given here [FMS96, p. 119]. With these, we can rewrite the Ward identities associated to translations, dilatations and Lorentz transforms as

$$\begin{aligned}
2\pi\partial_z\langle T_{\bar{z}z}X\rangle + 2\pi\partial_{\bar{z}}\langle T_{zz}X\rangle &= -\sum_{k=1}^{n}\partial_{\bar{z}}\frac{1}{z-w_k}\partial_{w_k}\langle X\rangle \\
2\pi\partial_z\langle T_{\bar{z}\bar{z}}X\rangle + 2\pi\partial_{\bar{z}}\langle T_{z\bar{z}}X\rangle &= -\sum_{k=1}^{n}\partial_z\frac{1}{\bar{z}-\bar{w}_k}\partial_{\bar{w}_k}\langle X\rangle \\
2\langle T_{z\bar{z}}X\rangle + 2\langle T_{\bar{z}z}X\rangle &= -\sum_{k=1}^{n}\delta(x-x_k)\Delta_k\langle X\rangle \\
-2\langle T_{z\bar{z}}X\rangle + 2\langle T_{\bar{z}z}X\rangle &= -\sum_{k=1}^{n}\delta(x-x_k)s_k\langle X\rangle
\end{aligned} \tag{3.52}$$

Adding and substracting the 2 last equations of (3.52) and using (3.51), we have

$$\begin{aligned}
2\pi\langle T_{\bar{z}z}X\rangle &= -\sum_{k=1}^{n}\partial_{\bar{z}}\frac{1}{z-w_k}h_k\langle X\rangle \\
2\pi\langle T_{z\bar{z}}X\rangle &= -\sum_{k=1}^{n}\partial_z\frac{1}{\bar{z}-\bar{w}_k}\bar{h}_k\langle X\rangle
\end{aligned} \tag{3.53}$$

Notice how in 2 dimensions, we naturally got back the possibility of having 2 different decoupled scaling dimensions due to the spin, as we saw in the paragraph following (3.16).

We renormalize the energy-momentum tensor and define the holomorphic and antiholomorphic energy-momentum fields as follows

$$T = -2\pi T_{zz} \qquad \bar{T} = -2\pi T_{\bar{z}\bar{z}} \tag{3.54}$$

Introducing this in the first 2 equations of (3.52), we have

$$\begin{aligned}\langle T(z)X\rangle &= \sum_{i=1}^{n}\left(\frac{1}{z-w_k}\partial_{w_k}\langle X\rangle + \frac{h_i}{(z-w_k)^2}\langle X\rangle\right) + \text{reg.}\\ \langle \bar{T}(z)X\rangle &= \sum_{i=1}^{n}\left(\frac{1}{\bar{z}-\bar{w}_k}\partial_{\bar{w}_k}\langle X\rangle + \frac{\bar{h}_i}{(\bar{z}-\bar{w}_k)^2}\langle X\rangle\right) + \text{reg.}\end{aligned} \tag{3.55}$$

where reg. stands for an holomorphic function of z or $\bar{z}$, regular when $z \to w_k$ or $\bar{z} \to \bar{w}_k$. Notice how the two newly defined energy-momentum fields decouple, with one depending only on the holomorphic dimension and the other depending only on the antiholomorphic dimension, thus explaining the name given to these fields. This shows once again the deep nature of the holomorphic and antiholomorphic parts of a CFT.

To end this subsection, we would like to bring the Ward identities associated to translations, Lorentz transformations and dilatations into a single Ward identity. We will not go into the details of the computation, but only go through the main idea of the proof. More details of this proof can be found here [FMS96, p. 121]. Let $x^\nu \to x^\nu + \epsilon^\nu(x)$ be an arbitrary conformal coordinate variation. We can write

$$\begin{aligned}\partial_\mu(\epsilon_\nu T^{\mu\nu}) &= \epsilon_\nu\partial_\mu T^{\mu\nu} + \frac{1}{2}(\partial_\nu\epsilon_\mu + \partial_\mu\epsilon_\nu)T^{\mu\nu} + \frac{1}{2}(\partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu)T^{\mu\nu}\\ &= \epsilon_\nu\partial_\mu T^{\mu\nu} + \frac{1}{2}(\partial_\rho\epsilon^\rho)\eta_{\mu\nu}T^{\mu\nu} + \frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha\epsilon_\beta\varepsilon_{\mu\nu}T^{\mu\nu}\end{aligned} \tag{3.56}$$

with $\varepsilon_{\mu\nu}$ the antisymmetric tensor. Using the 3 Ward identities on both sides, we have

$$\delta_\epsilon\langle X\rangle = \int \mathrm{d}^n w\mathrm{d}^2x\ \partial_\mu\langle T^{\mu\nu}(x)\epsilon_\nu(x)X\rangle \tag{3.57}$$

where the integral goes through all possible positions for the fields in the sequence X. Applying Gauss's theorem and using the notations introduced by (3.54), we finally get what is known as the *conformal Ward identity*:

$$\delta_{\epsilon,\bar{\epsilon}}\langle X\rangle = -\frac{1}{2\pi i}\oint_C \mathrm{d}z\ \epsilon(z)\langle T(z)X\rangle + \frac{1}{2\pi i}\oint_C \mathrm{d}\bar{z}\ \bar{\epsilon}(\bar{z})\langle\bar{T}(\bar{z})X\rangle \tag{3.58}$$

This single identity summarize the constraints of global conformal transformations[1]. From this, we will derive once again the generators of the conformal transformations, and get as announced earlier the Virasoro algebra.

3.7 Operator product and mode expansions

In this subsection, we will introduce some notations to have a better way to deal with correlation functions, states and operators.

Keen eyes might have noticed that in the correlation function of the free boson (2.78) as well as in the Ward identities (3.55), the correlation function diverges when $z \to w_k$ or $\bar{z} \to \bar{w}_k$. It is very typical of correlation functions to

[1] the constraints associated to the special conformal transformation is taken into account by making the energy-momentum traceless

diverge when the position of two fields coincide. To understand this, recall that for 2 fields φ_1, φ_2, the correlation function $\langle \varphi_1(x_1)\varphi_2(x_2)\rangle$ represents some kind of probability of the state $\varphi_2(x_2)|0\rangle$ ending as the state $\varphi_1^\dagger(x_1)|0\rangle$. More precisely, using the translation and rotation invariance, we see that $\langle \varphi_1(x_1)\varphi_2(x_2)\rangle$ is a function of $|x_1 - x_2|$, and can be seen as a probability density depending on the distance of the two local operators. But then, we can expect the probability density to be concentrated around 0, as two events closer in spacetime have more chance to be correlated. We therefore expect the function $\langle \varphi_1(x_1)\varphi_2(x_2)\rangle$ to diverge as $|x_1 - x_2|$ tends to 0, as a Dirac function would.

We should therefore always expect correlation functions to diverge like that. More specifically, it is always this divergence that interests us. Taking the notations from the paragraph above, most of the density of the probability is located near $|x_1 - x_2| \sim 0$. We are not interested in the small residues of the probability for medium distances of $|x_1 - x_2|$, but in the main correlation near $|x_1 - x_2| \sim 0$. For this reason, we will almost always study correlation functions depending on $|x_1 - x_2|$, and expand the function in its Laurent serie for $|x_1 - x_2| \sim 0$, as to get the exact speed and factors of the divergence. Moreover, we will usually not be interested in the regular terms, but only in the diverging terms, which come in finite numbers. In general, we can write each of these terms as an operators well defined when $z \to w$, multiplied by a function diverging when $z \to w$. We call this representation the operator product expansion, or OPE. We usually write it by removing the brackets $\langle \dots \rangle$, without forgetting that it only makes sens inside of correlation functions.

For exemple, the Ward identities (3.55) can be rewritten for a primary field of conformal dimensions $h, \bar{h}$ in term of OPE as

$$\begin{aligned} T(z)\varphi(w,\bar{w}) &\sim \frac{h}{(z-w)^2}\varphi(w,\bar{w}) + \frac{1}{z-w}\partial_w\varphi(w,\bar{w}) \\ \bar{T}(\bar{z})\varphi(w,\bar{w}) &\sim \frac{h}{(\bar{z}-\bar{w})^2}\varphi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\varphi(w,\bar{w}) \end{aligned} \tag{3.59}$$

We immediatly see that this way of writing correlation functions is way more convenient and allow us to quickly see what is important.

Remark. The OPE equips the space of operators with a sort of algebraic structure. This is exactly the structure we will define and study on the mathematical side. Understanding the mathematical structure behind a CFT allows us to know the OPE of every fields, and therefore to compute any correlation function, effectively solving the theory. This explains the huge importance the algebra of operators has in the study of CFTs.

Now that we have seen how to conveniently write correlation functions, we would like to conveniently write fields in general. After that, we will be able to link the 2 to obtain a nice relation which will later enable us to finally find the generators of conformal transformations present in the space of operators.

The idea to have a better formulation of fields is to expand fields of the form $\varphi(z,\bar{z})$ with conformal dimensions $(h,\bar{h})$ in terms of a family of operators independant of the position. Thus, let $\varphi(z,\bar{z})$ be a field of our theory. As the

field is holomorphic in z and in $\bar{z}$, it may be written

$$\begin{aligned} \varphi(z,\bar{z}) &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} z^{-m-h}\bar{z}^{-n-\bar{h}}\varphi_{m,n} \\ \text{with} \qquad \varphi_{m,n} &= \frac{1}{2\pi i}\oint \mathrm{d}z\ z^{m+h-1}\frac{1}{2\pi i}\oint \mathrm{d}\bar{z}\ \bar{z}^{n+\bar{h}-1}\varphi(z,\bar{z}) \end{aligned} \tag{3.60}$$

In this definition, we shift the powers of z and $\bar{z}$ by h and $\bar{h}$ respectively, such that if we compare the straightforward Hermitian conjugation of this expression with the Hermitian conjugation defined by (3.33), we get

$$\varphi^\dagger_{m,n} = \varphi_{-m,-n} \tag{3.61}$$

We will now drop the antiholomorphic coordinate to simplify the notations. We must though not forget that it is always here, and can easily be restored. We have

$$\begin{aligned} \varphi(z) &= \sum_{m\in\mathbb{Z}} z^{-m-h}\varphi_m \\ \varphi_m &= \frac{1}{2\pi i}\oint \mathrm{d}z\ z^{m+h-1}\varphi(z) \end{aligned} \tag{3.62}$$

With this, let's relate contour integrals to commutators in the operator product expansions. Let $a(z)$ and $b(z)$ be two holomorphic fields. We consider the integral

$$\oint_w \mathrm{d}z\ a(z)b(w) \tag{3.63}$$

where the integral goes around w without getting around 0. This expression as a meaning inside of correlation functions, as long as it is time ordered. We can write

$$\oint_w \mathrm{d}z\ a(z)b(w) = \oint_{C_1} \mathrm{d}z\ a(z)b(w) - \oint_{C_2} \mathrm{d}z\ b(w)a(z) \tag{3.64}$$

where the two integrals on the right hand side are integrals at fixed time, meaning C_1 and C_2 are 2 circles centered around 0, and where C_1 wraps around w and C_2 doesn't.

Remark. Inside a correlation function, there are usually more fields than these two. This relation is true if the only field having a singularity inside of C_1 and not C_2 is $b(w)$. We should thus choose C_1 and C_2 as close as possible.

Defining the operator

$$A \equiv \oint a(z)\mathrm{d}z \tag{3.65}$$

we then have

$$\oint_w \mathrm{d}z\ a(z)b(w) = [A, b(w)] \tag{3.66}$$

Note that with the notations of (3.62), we have $A = a_{1-h}$, where h is the holomorphic dimension of a.

Defining the operator B similarly to A, we can once again integrate to obtain

$$[A,B] = \oint_0 \mathrm{d}w \oint_w \mathrm{d}z\ a(z)b(w) \tag{3.67}$$

This relation is strong, as it allows one to explicitely compute commutators as integrals, changing the landscape from unknown algebras to well-known analysis.

3.8 The Virasoro algebra

We will now showcase an application of the relation (3.67) we just computed, to the energy-momentum tensor.

The energy-momentum field is a field inside of our theory, conveying some sort of data about the conformal symmetry of our theory. Indeed, it is originally defined as a 2-tensor giving the conserved current for translations. But it is made symmetric thanks to the Lorentz invariance, and made traceless thanks to the scale invariance and special conformal transformation invariance. Therefore, using all of the conformal symmetries, we obtained 2 fields, one depending only on the holomorphic dimension and the other depending only on the antiholomorphic dimension. We may expect that the modes of both fields contain information on the symmetry. More explicitely, knowing from section 3.5 that the representation of the conformal transformations decouples into an holomorphic and an antiholomorphic part, knowing that integrating a current leads to some sort of generator and that the energy-momentum fields act as some sort of currents for the conformal symmetry, we can hope that the modes of the energy-momentum fields will generate the conformal transformations. We will now proceed to proving it.

We should first concentrate on the holomorphic dimension, knowing we can do the other side similarly. As was done to get the conformal Ward identity (3.58), we let $\epsilon(z)$ be the holomorphic component of an infinitesimal conformal transformation. Defining the conformal charge

$$Q_\epsilon \equiv \frac{1}{2\pi i}\oint \mathrm{d}z\epsilon(z)T(z) \tag{3.68}$$

we can rewrite the conformal Ward identity using (3.66) as follows

$$\delta_\epsilon\varphi(w) = -[Q_\epsilon, \varphi(w)] \tag{3.69}$$

We see here that the conformal charge acts as a generator for conformal transformations, in a similar way to generators defined by (2.31).

Using the mode expansion defined by (3.62), we can write

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n \tag{3.70}$$

Also writing

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}\epsilon_n \tag{3.71}$$

we can rewrite (3.68) as

$$Q_\epsilon = \sum_{n\in\mathbb{Z}} \epsilon_n L_n \tag{3.72}$$

Remark. We have the exact same equations in the antiholomorphic dimension, defining the operators $(\bar{L}_n)_n$.

This shows that the mode operators $(L_n)_n$ and $(\bar{L}_n)_n$ generate the conformal transformations on the Hilbert space. In particular, we see that L_0 and $\bar{L}_0$ generate the dilatations. But in radial quantization, dilatations correspond to time translations. As such, we see that the Hamiltonian H of our system verifies

$$H \propto L_0 + \bar{L}_0 \tag{3.73}$$

It is possible to compute the commutation relation of the mode operators $(L_n)_n$ and $(\bar{L}_n)_n$, though it is highly non-trivial and almost nowhere to be found in popular physics books. We will only give the idea of the proof. First, we need to consider a primary field $\varphi(z, \bar{z})$. We then want to expand the identity

$$[[Q_{\epsilon_2}, Q_{\epsilon_1}], \varphi(z, \bar{z})] = [Q_{\epsilon_2}, [Q_{\epsilon_1}, \varphi(z, \bar{z})]] - [Q_{\epsilon_1}, [Q_{\epsilon_2}, \varphi(z, \bar{z})]] \tag{3.74}$$

In this, we can compute the right-hand side using (3.59), as to get the expression of the OPE of the energy-momentum field with itself. When deriving the expression, we have the freedom to add or not a term, up to a constant. We should fix this constant to be C. We get

$$T(z)T(w) \sim \frac{C}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{3.75}$$

Classically, we define $c = 2C$ the central charge of the theory. The OPE of the energy-momentum field with itself rewrites

$$T(z)T(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{3.76}$$

Similarly, in the antiholomorphic dimension, we have

$$\bar{T}(\bar{z})\bar{T}(\bar{w}) \sim \frac{\frac{c}{2}}{(\bar{z}-\bar{w})^4} + \frac{2\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{T}(\bar{w})}{\bar{z}-\bar{w}} \tag{3.77}$$

Remark. From this OPE, recalling 3.59, we see that the energy-momentum fields are not primary fields. However, we see that the holomorphic energy-momentum field has conformal dimensions $(2, 0)$, while the antiholomorphic one has conformal dimensions $(0, 2)$. This justifies the shift by 2 in the definition of the modes (3.70).

From this, using the relation between commutators in OPEs and contour integrals (3.67) and the definition of the families $(L_n)_n$ and $(\bar{L}_n)_n$ (3.70), we can finally compute the commutation relation of the $(L_n)_n$ and $(\bar{L}_n)_n$:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m=-n} \\
[\bar{L}_m, L_n] &= 0 \\
[\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m=-n}
\end{aligned} \tag{3.78}$$

The non-linear Lie algebra composed of $(L_n)_n$, or equivalently $(\bar{L}_n)_n$, is called the Virasoro algebra, and depends on the central charge c. We write it $\mathrm{Vir}(c)$.

The complete (non-chiral) Hilbert space thus possess a representation of $\mathrm{Vir}(c) \times \overline{\mathrm{Vir}}(c)$.

The Virasoro algebra is the unique central extension of the Witt algebra derived in 3.5, which shows that the energy-momentum tensor indeed generates a representation of the conformal transformations in 2 dimensions. In particular, recalling from 3.2 that the conformal group in 2 dimensions is isomorphic to $SL(2,\mathbb{C}) \times \overline{SL(2,\mathbb{C})}$, we see that we indeed have a representation of the conformal group inside $\mathrm{Vir}(c) \times \overline{\mathrm{Vir}}(c)$, generated by $L_{-1}, L_0, L_1, \bar{L}_{-1}, \bar{L}_0$, and $\bar{L}_1$. In particular, $L_0 + \bar{L}_0$ generates dilatations (explaining the nomenclature of scaling dimension), $L_0 - \bar{L}_0$ generates rotations (explaining the nomenclature of spin), and L_{-1} and $\bar{L}_{-1}$ both generate translations.

Remark. It is important to remember that the Virasoro algebra we derived generate the (local) conformal transformations, which are not symmetries of the system. In particular, they do not commute with the Hamiltonian $(L_0 + \bar{L}_0)$.

Now, what is the central charge c, apparently totally arbitrary? Some theories are well defined for any value of c, while some others are well defined only for discrete values of c. We call the first class of CFT *classical* CFTs, while we call the second class *exotic* CFTs.

We should interpret c as a default in the conformal symmetry. Indeed, we have shown that the true algebra corresponding to conformal transformations is the Witt algebra, that is the Virasoro algebra for $c = 0$. But in reality, the conformal symmetry is rarely a real symmetry of the system. In the exemple of condensed matter physics and phase transitions for exemple, the conformal symmetry is only valid at a mesoscopic term. Thus, if we continue to renormalize the system, we will hit a wall at some point. c is some sort of quantity representing this anomaly in the theory, and reminding us that conformal field theories (and quantum field theories in general) are never anything more than a framework, working only at a certain scale, used to describe reality only up to some point.

3.9 The Hilbert space

Now that we have derived the Virasoro algebra, the actual algebra of operators present inside our space of operators and representing the conformal transformations, we can try to understand the form of the space of states $\mathcal{H}$.

Let's consider a 2D conformal field theory. We will assume that the theory is unitary (only allowing states with positive norms) since this is the case of most of the 2D CFTs appearing in nature. As we have just seen, the energy-momentum tensor decouples into two currents T and $\bar{T}$ whose modes form two Virasoro algebras $(L_n)_n$ and $(\bar{L}_n)_n$. These operators are therefore part of the set of operators in the CFT, and form representations of the Virasoro algebra on the Hilbert space. Let's concentrate on representations of the Virasoro algebra in the holomorphic dimension, coming from the modes of T.

Let $(|*\rangle_n)_n \subset \mathcal{H}$ a representation of the Virasoro algebra which is a part of the Hilbert space. As we are dealing with physical states, we can expect their energy to be bounded from below. Indeed, for any set of states included in the

Hilbert space, there should always be a state with minimal energy, corresponding to some sort of "vacuum" state[2]. But the energy of a state is given by the Hamiltonian $L_0 + \bar{L}_0$, and the spin given by $L_0 - \bar{L}_0$ should also be bounded, resulting in the fact that the eigenvalues of L_0 should be bounded from below. There should therefore be a primary state $|h\rangle$ in the representation such that this state has the lowest eigenvalue for L_0, which we will call its conformal (holomorphic) dimension and write h. Let's try to see how the Virasoro algebra act on this state.

We know that $[L_0, L_m] = -mL_m$. Therefore, for a general state $|e\rangle$ such that

$$L_0|e\rangle = e|e\rangle \tag{3.79}$$

for $e \in \mathbb{R}$, we have for $m \in \mathbb{Z}$

$$\begin{aligned} L_0 L_m|e\rangle &= ([L_0, L_m] + L_m L_0)|e\rangle \\ &= (e - m)L_m|e\rangle \end{aligned} \tag{3.80}$$

As such, L_m lowers the energy of $|e\rangle$ if $m > 0$, and raises its energy if $m < 0$. As an immediate corollary,

$$L_m|h\rangle = 0 \quad \forall m > 0 \tag{3.81}$$

Remark. This equation precisely defines what is called a primary state. As the name imply, the field corresponding to a primary state is a primary field, and conversely the state corresponding to a primary field is a primary state. This explains the importance of primary fields, as any representation of the Virasoro algebra has for lowest energy state a primary state.

We can then recognize $(|*\rangle_n)_n$ as a highest weight representation of the Virasoro algebra with primary state $|h\rangle$, where the $(L_{-m})_{m>0}$ act as raising operators whilst the $(L_m)_{m>0}$ act as lowering operators. The representation is spanned by states of the form

$$(L_{-n_1})^{r_1}(L_{-n_2})^{r_2} \ldots |h\rangle \tag{3.82}$$

where $n_1 > n_2 > \ldots$ and $r_1, r_2, \ldots$ are integers. Such a representation is determined up to an isomorphism by the couple (h, c), where c is the central charge of the Virasoro algebra. We usually write it $V(h, c)$.

We know that the Hilbert space of the conformal field theory is made up of representation(s) of the Virasoro algebra. But we have just seen that the representations appearing in the Hilbert space are highest weight representations, which are determined up to isomorphism by the central charge of the theory and by the conformal dimension of the highest weight vector. Therefore, we may write the Hilbert space as

$$\mathcal{H} = \bigoplus_{a,b} V(h_a, c) \otimes \overline{V}(\bar{h}_b, c) \tag{3.83}$$

where a indexes the holomorphic dimensions of primary fields, and b indexes the antiholomorphic dimensions.

[2]We quote the word "vacuum" as it may not be the actual vacuum state, especially in sectors with non-trivial monodromy

Remark. We have written $\otimes$ as the operator linking the holomorphic and antiholomorphic Verma modules. This symbol should be taken as a symbol more than anything, as in general all of the product states won't be present in $\mathcal{H}$. In particular, some CFTs are *diagonal*, meaning they take the form

$$\mathcal{H} = \bigoplus_a V(h_a, c) \otimes \overline{V}(\bar{h}_a, c) \tag{3.84}$$

From these, we see that as announced in 3.2, the entire space of states itself splits into the product of holomorphic and antiholomorphic states. What is the reason for this splitting? One might find an explanation when considering the relativistic behaviour of particles. A CFT usually models some kind of particles, seen as the excitations of the fields. But as mentionned in 3.1, the notion of mass cannot exist in a CFT. Therefore, the modelled particles should always be massless. But we know that in general, for a particle,

$$E^2 = m^2c^4 + p^2c^2 \tag{3.85}$$

So for a massless particle, we have $E = pc$, but the velocity of the particle is given by $v = E/P$ such that we have

$$v = c \tag{3.86}$$

meaning the particles modelled by a CFT should always travel at the speed of light through the space dimension.

However, the space only has dimension 1. Obviously, a particle cannot continuously do a U-turn and keep its speed of light in 1 dimension. Therefore, a particles should either always and forever move left, or move right. This leads to the decomposition in the left and right moving sectors, or in the holomorphic and antiholomorphic sectors.

The fact that the Hilbert space splits in 2 then makes sense. A general state should be composed of particles moving left, and particles moving right, both non-interacting. This also explains the decomposition of the energy-momentum tensor in two energy-momentum fields, corresponding to the total momentum of the particles moving left, and total momentum of the particles moving right.

3.10 The free boson, part 2

Now that we have the basic tools to study a CFT, we can return to the exemple of the free boson, defined in 2.7. Indeed, we can easily verify that the action of the free boson (2.61) is invariant under conformal transformations. In particular, we now have the tools to prove that the energy-momentum defined by (2.79) is really the energy-momentum tensor. But first, let's return to the field φ.

We have previously compute the correlation function of φ with itself 2.78. In the chiral coordinates, using the notations of OPEs, this rewrites

$$\varphi(z, \bar{z})\varphi(w, \bar{w}) \sim \frac{1}{4\pi}(\ln(z - w) + \ln(\bar{z} - \bar{w})) \tag{3.87}$$

This OPE does not look very natural. However, it should be expected as φ itself is not so natural, and in particular is not a primary field, as we will see later on.

However, taking the derivatives $\partial_z\varphi$ and $\partial_{\bar{z}}\varphi$, we can separate the holomorphic and antiholomorphic coordinates

$$\begin{aligned}\partial_z\varphi(z,\bar{z})\partial_w\varphi(w,\bar{w}) &\sim \frac{1}{4\pi}\frac{1}{(z-w)^2}\\ \partial_{\bar{z}}\varphi(z,\bar{z})\partial_{\bar{w}}\varphi(w,\bar{w}) &\sim \frac{1}{4\pi}\frac{1}{(\bar{z}-\bar{w})^2}\end{aligned} \tag{3.88}$$

Here, the fact that the two variables z and w are interchangeable show the interchangeability of the fields, and thus the bosonic nature of the fields. The decoupling of the holomorphic and antiholomorphic dimension in the first derivative of φ should be expected, as it computes some sort of momentum of the left and right moving particles.

Now, let's go back to the energy-momentum tensor. We want to prove that the tensor $T_{\mu\nu}$ defined by

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi \tag{3.89}$$

is the energy-momentum tensor of the free boson theory. Intuitively, this can be more or less expected. Indeed, the momentum-energy tensor is supposed to give the energy of the field. But the field describes free particles, and the energy of a free particle is given solely by its kinetic energy as it is free. The kinetic energy of a particle is classically

$$E_k = \frac{p^2}{2m} \propto (\partial_t x)^2 \tag{3.90}$$

which looks like (3.89). This is even more evident in complex coordinates, where the two off-diagonal terms vanish, leaving the 2 decoupled fields

$$\begin{aligned}T(z) &= -2\pi : \partial\varphi(z)\partial\varphi(z) :\\ \overline{T}(\bar{z}) &= -2\pi : \bar{\partial}\varphi(\bar{z})\bar{\partial}\varphi(\bar{z}) :\end{aligned} \tag{3.91}$$

Here, we clearly see the expression of the kinetic energy of left moving and right moving particles, as discussed in the previous subsection. Note that the normal ordering naturally appear here, showing the quantum nature of the theory.

We concentrate on the holomorphic dimension, knowing the same happens in the antiholomorphic one. We can calculate the OPE of $T(z)$ with $\partial\varphi$ using Wick's theorem, knowing the OPE of $\partial\varphi$ with itself:

$$\begin{aligned}T(z)\partial\varphi(w) &= -2\pi : \partial\varphi(z)\partial\varphi(z) : \partial\varphi(w)\\ &\sim -2\pi : \partial\varphi(z)\partial\overbrace{\varphi(z) : \partial}\varphi(w) - 2\pi : \partial\overbrace{\varphi(z)\partial\varphi(z) : \partial}\varphi(w) : \partial\varphi(w)\\ &\sim \frac{\partial\varphi(z)}{(z-w)^2}\\ &\sim \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial_w^2\varphi(w)}{z-w}\end{aligned} \tag{3.92}$$

Recalling (3.59), this is already a good indication of the fact that $T^{\nu\mu}$ indeed is the energy-momentum of the theory. Moreover, if it really is the energy-momentum tensor of the theory, this shows that $\partial\varphi$ is a primary field of conformal dimension $h = 1$.

Following the same methods, we can also compute the OPE of the energy-momentum tensor with itself:

$$T(z)T(w) \sim \frac{1}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{3.93}$$

This OPE exactly corresponds to the one we derived in (3.76), from which one shows that the modes of this field form the Virasoro algebra. Therefore, the modes of $T(z)$ and $\overline{T}(\bar{z})$ both form a Virasoro algebra, definitely proving that they are the energy-momentum fields of the theory.

Note that from this OPE, we also find that the central charge of the free boson theory is equal to 1.

3.11 Radial quantization for the free boson

Now we would like to use radial quantization to our advantage to compute even more things on the theory of the free boson. In particular, since we have set the exact dimensions associated to time and space, we should be able to derive explicit expressions for a lot of things.

To do so, we should first study the theory on a cylinder, as we compactify the space. We write L the circumference of the cylinder. The theory stays the same, except the fact that we add a periodicity condition on the field

$$\varphi(x+L,t) = \varphi(x,t) \tag{3.94}$$

As the field is periodic, we can expand it using the Fourier transform

$$\begin{aligned} \varphi(x,t) &= \sum_n e^{2\pi i n x/L} \varphi_n(t) \\ \text{with} \quad \varphi_n(t) &= \frac{1}{L}\int \mathrm{d}x \; e^{-2\pi i n x/L} \varphi(x,t) \end{aligned} \tag{3.95}$$

Introducing the Fourier modes into the Lagrangian (2.61), we get

$$\mathcal{L} = \frac{1}{2} L \sum_n \left(\dot{\varphi}_n \dot{\varphi}_{-n} - \left(\frac{2\pi n}{L}\right)^2 \varphi_n \varphi_{-n} \right) \tag{3.96}$$

Now, the momentum associated to ϕ_n is

$$\pi_n = L\dot{\varphi}_{-n} \tag{3.97}$$

We do still have $[\varphi_n, \pi_m] = i\delta_{n,m}$. We can write the Hamiltonian

$$H = \frac{1}{2L}\sum_n (\pi_n \pi_{-n} + (2\pi n)^2 \varphi_n \varphi_{-n}) \tag{3.98}$$

We should highlight that

$$\varphi_n^\dagger = \varphi_{-n} \qquad \pi_n^\dagger = \pi_{-n} \tag{3.99}$$

The Hamiltonian still corresponds to a sum of decoupled harmonic oscillators, of frequencies $\omega_n = \frac{2\pi|n|}{L}$. This is expected, since we haven't modified the theory.

We see that the term $n = 0$ vanishes. This is due to the absence of mass of the system, and is the cause of its conformal invariance. With a mass, the system wouldn't be invariant through conformal transformations.

Let's now define the creation $\tilde{a}_n^\dagger$ and annihilation $\tilde{a}_n$ operators. Usually, one would define them as follows

$$\tilde{a}_n = \frac{1}{\sqrt{4\pi|n|}}(2\pi|n|\varphi_n + i\pi_{-n}) \tag{3.100}$$

so that we can have the usual $[\tilde{a}_n, \tilde{a}_m] = 0, [\tilde{a}_n\tilde{a}_m^\dagger] = \delta_{n,m}$. However, this definition does not work with the zero mode.

Instead, we will treat the zero mode separately, and define the following operators

$$\begin{aligned} &\text{for } n > 0, \quad a_n = -i\sqrt{n}\tilde{a}_n \quad \text{and} \quad \bar{a}_n = -i\sqrt{n}\tilde{a}_{-n} \\ &\text{for } n < 0, \quad a_n = -i\sqrt{n}\tilde{a}_{-n}^\dagger \quad \text{and} \quad \bar{a}_n = -i\sqrt{n}\tilde{a}_n^\dagger \end{aligned} \tag{3.101}$$

These operators have the following commutation relations

$$[a_n, a_m] = n\delta_{n,m} \qquad [a_n, \bar{a}_m] = 0 \qquad [\bar{a}_n, \bar{a}_m] = n\delta_{n,m} \tag{3.102}$$

Writing the Hamiltonian in term of these operators, we have

$$H = \frac{1}{2L}\pi_0^2 + \frac{2\pi}{L}\sum_{n\geq 0}(a_{-n}a_n + \bar{a}_{-n}\bar{a}_n) \tag{3.103}$$

Note how this Hamiltonian takes a quite explicit form, as it is a sum on each kind of excitation (particle) of their density operator. (3.102) leads to

$$[H, a_{-n}] = \frac{2\pi}{L}na_{-n} \tag{3.104}$$

which means that for $|E\rangle$ an eigenstate of H of energy E, $a_{-n}|E\rangle$ is still an eigenstate of H with energy $E + \frac{2n\pi}{L}$

Expressing the Fourier modes according to these operators, we have

$$\varphi_n = \frac{i}{n\sqrt{4\pi}}(a_n - \bar{a}_{-n}) \tag{3.105}$$

We can thus write at $t = 0$

$$\varphi(x) = \varphi_0 + \frac{i}{\sqrt{4\pi}}\sum_{n\neq 0}\frac{1}{n}(a_n - \bar{a}_{-n})e^{2\pi inx/L} \tag{3.106}$$

But then, thanks to the explicit expression of the Hamiltonian (3.103), we have

$$\varphi(x,t) = \varphi_0 + \frac{1}{L}\pi_0 t + \frac{i}{\sqrt{4\pi}}\sum_{n\neq 0}\frac{1}{n}\left(a_n e^{2\pi in(x-t)/L} - \bar{a}_{-n}e^{2\pi in(x+t)/L}\right) \tag{3.107}$$

Going to the Euclidian spacetime by replacing t with $-i\tau$ and using the complex coordinates

$$z = e^{2\pi(\tau - ix)/L} \qquad \bar{z} = e^{2\pi(\tau + ix)/L} \tag{3.108}$$

we finally have

$$\varphi(z,\bar{z}) = \phi_0 - \frac{i}{4\pi}\pi_0 \ln(z\bar{z}) + \frac{i}{\sqrt{4\pi}} \sum_{n\neq 0} \frac{1}{n}(a_n z^{-n} + \bar{a}_n \bar{z}^{-n}) \tag{3.109}$$

We can explicitely see this time that φ is not a primary field. But we know from (3.92) that its derivatives $\partial\varphi$ and $\bar{\partial}\bar{\varphi}$ are. We may therefore also compute them explicitely. Let's concentrate on the holomorphic derivative.

Derivating the expansion of φ found just above, we have

$$i\partial\varphi(z) = \frac{\pi_0}{4\pi z} + \frac{1}{\sqrt{4\pi}} \sum_{n\neq 0} a_n z^{-n-1} \tag{3.110}$$

Introducing

$$a_0 \equiv \bar{a}_0 \equiv \frac{\pi_0}{\sqrt{4\pi}} \tag{3.111}$$

we can include the zero term in our sum, and write

$$i\partial\varphi(z) = \frac{1}{\sqrt{4\pi}} \sum_{n} a_n z^{-n-1} \tag{3.112}$$

Now that we have an explicit expression of $\partial\varphi$ we can also explicitely express $T(z)$. Recalling (3.91), we have

$$T(z) = \frac{1}{2} \sum_{n,m\in\mathbb{Z}} z^{-n-m-2} : a_n a_m : \tag{3.113}$$

which implies

$$\begin{aligned} L_n &= \frac{1}{2} \sum_{m\in\mathbb{Z}} a_{n-m} a_m \qquad \text{for } n \neq 0 \\ L_0 &= \sum_{n\geq 0} a_{-n} a_n + \frac{1}{2} a_0^2 \end{aligned} \tag{3.114}$$

We can then rewrite (3.103) as

$$H = \frac{2\pi}{L}(L_0 + \bar{L}_0) \tag{3.115}$$

which is the result predicted in 3.4.

Now, let's look at the rest of our theory. Are there other primary fields in the theory, which could generate more Verma modules in $\mathcal{H}$? The energy momentum tensor and the partial derivatives of the field φ currently make an algebra closed under the OPE, thus making a consistent theory. However, we may be able to extend it using other primary fields.

The key observation is to see that since $\partial\varphi$ has a scaling dimension of 1, φ has a vanishing scaling dimension. Making use of this, we can define a family of fields $(\mathcal{V}_\alpha)_{\alpha\in\mathbb{R}}$ without introducing any notion of scale, which we call the vertex operators:

$$\begin{aligned} \mathcal{V}_\alpha(z,\bar{z}) &\equiv : e^{i\alpha\phi(z,\bar{z})} : \\ &= e^{i\alpha\varphi_0 + \frac{\alpha}{\sqrt{4\pi}} \sum_{n\geq 0} \frac{1}{n}(a_{-n} z^n + \bar{a}_{-n} \bar{z}^n)} e^{\frac{\alpha}{4\pi}\pi_0 \ln(z\bar{z}) - \frac{\alpha}{\sqrt{4\pi}} \sum_{n\geq 0} \frac{1}{n}(a_n z^{-n} + \bar{a}_n \bar{z}^{-n})} \end{aligned} \tag{3.116}$$

We want to see how these newly defined fields act. Let's first calculate the OPE of $\partial\varphi$ with $\mathcal{V}_\alpha$, using the definition of the exponential and Wick's theorem, as was done to compute (3.92):

$$\begin{aligned}\partial\varphi(z)\mathcal{V}_\alpha(w,\bar{w}) &= \sum_{n=0}^{+\infty}\frac{(i\alpha)^n}{n!}\partial\varphi(z):\varphi(w,\bar{w})^n: \\ &\sim \frac{-1}{4\pi(z-w)}\sum_{n=1}^{+\infty}\frac{(i\alpha)^n}{(n-1)!}:\varphi(w,\bar{w})^{n-1}: \\ &\sim \frac{i\alpha}{4\pi}\frac{\mathcal{V}_\alpha(w,\bar{w})}{z-w}\end{aligned} \tag{3.117}$$

Using this, we compute its OPE with the energy-momentum tensor

$$\begin{aligned}T(z)\mathcal{V}_\alpha(w,\bar{w}) &= -2\pi\sum_{n=0}^{+\infty}\frac{(i\alpha)^n}{n!}:\partial\varphi(z)\partial\varphi(z)::\varphi(w,\bar{w})^n: \\ &\sim \frac{1}{8\pi(z-w)^2}\sum_{n=2}^{+\infty}\frac{(i\alpha)^n}{(n-2)!}:\varphi(w,\bar{w})^{n-2}: \\ &\quad + \frac{1}{z-w}\sum_{n=1}^{+\infty}\frac{(i\alpha)^n}{n!}n:\partial\varphi(z)\varphi(w,\bar{w})^{n-1}: \\ &\sim \frac{\alpha^2}{8\pi}\frac{\mathcal{V}_\alpha(w,\bar{w})}{(z-w)^2}+\frac{\partial_w\mathcal{V}_\alpha(w,\bar{w})}{z-w}\end{aligned} \tag{3.118}$$

The 2 terms here comes from the single and double contractions in the application of Wick's theorem. Thanks to this OPE, we see that the fields $\mathcal{V}_\alpha$ are primary fields, of holomorphic dimension

$$h(\alpha)=\frac{\alpha^2}{8\pi} \tag{3.119}$$

Knowing that the OPE of $\mathcal{V}_\alpha$ with the antiholomorphic energy-momentum tensor is of the same form, we also get

$$\bar{h}(\alpha)=h(\alpha)=\frac{\alpha^2}{8\pi} \tag{3.120}$$

In conclusion, we end up with a theory whose central charge is 1, and whose Hilbert space consists of the primary fields $\partial\varphi,\bar{\partial}\varphi,\mathcal{V}_\alpha$ for $\alpha\in\mathbb{R}$, and of the identity, together with their descendants under the Virasoro algebra.

Bibliography

[Cha18] Sourav Chatterjee. *Introduction to Quantum Field Theory for Mathematicians*. 2018. URL: https://souravchatterjee.su.domains/qft-lectures-combined.pdf.

[FMS96] Philippe Di Francesco, Pierre Mathieu, and David Sénéchal. *Conformal field theory*. 1st ed. Graduate Texts in Contemporary Physics. Springer New York, NY, Dec. 1996. ISBN: 978-0-387-94785-3. DOI: 10.1007/978-1-4612-2256-9.