

\mathcal{W} -SYMMETRY IN CONFORMAL FIELD THEORY

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Abstract

We review various aspects of \mathcal{W} -algebra symmetry in two-dimensional conformal field theory and string theory. We pay particular attention to the construction of \mathcal{W} -algebras through the quantum Drinfeld-Sokolov reduction and through the coset construction.

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1. Introduction

1.1. *Extensions of conformal symmetry*

Conformal invariance in two dimensions is a spectacularly powerful symmetry. Two-dimensional quantum field theories that possess conformal symmetry, which are called conformal field theories, can be solved exactly by exploiting the conformal symmetry. This fact, and the circumstance that conformal field theories have found remarkable applications in string theory (see [176]) and in the study of critical phenomena in statistical mechanics (see [208] for a collection of reprints), has resulted in a large-scale study of conformal field theories in recent years.

From the mathematical point of view, the main reason why conformal symmetry is so powerful, is the fact that the corresponding symmetry algebra, which is the product of two copies of the Virasoro algebra, is infinite-dimensional. In a quantum field theory the conformal symmetry gives rise to Ward identities that interrelate various correlation functions. In certain special theories (so-called minimal models) these relations take the form of differential equations whose solutions provide an explicit solution of the theory [29].

In string theory, conformal symmetry arises as a remnant of the reparametrization invariance of the string world-sheet, which guarantees that the physics of a string theory does not depend on the choice of coordinates on the world-sheet. Due to this, the conformal symmetry in a string theory is ‘gauged’, which implies that the physical states satisfy a number of constraints, the so-called Virasoro constraints, which can be compared to the Gauss-law in quantum electro-dynamics. In the modern formalism these constraints are implemented in a BRST quantization procedure. Consistency of the quantization requires that the conformal invariance is not affected at the quantum level, which leads to important conditions on the space-time backgrounds in which a quantum string can propagate. Through this mechanism, string theory makes contact with general relativity, thus raising the hope that quantum strings may teach us about a consistent theory of quantum gravity.

The applications of conformal field theory to statistical mechanics have led to exact results for critical exponents and finite-size corrections for statistical systems (two-dimensional classical lattice models or one-dimensional quantum chains) at a second order phase transition point. The classical example of the Ising model (corresponding to a con-

formal field theory of central charge $c = 1/2$) has been generalized in many directions, which has led to a wealth of systems for which the critical behavior is known exactly.

Since the early days of the massive attention for conformal field theory, several fields have been explored which are closely related to conformal field theory, and in which conformal field theory ‘technology’ is heavily used. For example, the study of so-called perturbed conformal field theories has given rise to surprising new results for certain massive integrable quantum field theories [344]. The study of two-dimensional gravity (pure or coupled to matter fields) relies heavily on conformal field theory techniques and the same is true for two-dimensional topological quantum field theories. These applications provide additional motivation for a detailed study of the structure of conformal field theories.

In a systematic study of $D = 2$ conformal quantum field theory extensions of the conformal symmetry play an important role. The algebraic structures that emerge in the study of bosonic extended symmetry are higher-spin extensions of the Virasoro algebra, which are commonly called \mathcal{W} -algebras. These algebras, and the associated \mathcal{W} -symmetry in conformal field theory, are the main topic of this review.

There are two main reasons for studying extended symmetries in conformal field theory. The first is that certain applications of conformal field theory (in string theory or statistical mechanics) require some extra symmetry in addition to conformal invariance. The second reason is that extended symmetries can be used to facilitate the analysis of a large class of conformal field theories (called rational conformal field theories) and, eventually, to classify certain types of conformal field theories. In the remainder of this introductory section we shall briefly discuss these two aspects.

We first take a look at the role played by extended symmetries in applications of conformal field theory. An important example of this are applications in string theory, where extensions of the worldsheet conformal symmetry have been very important. Probably the best known example for this is the $N = 1$ supersymmetric extension of conformal symmetry, called superconformal symmetry, which promotes strings to superstrings, thereby improving their properties. Compactified superstrings that are supersymmetric in space-time require $N = 2$ extended superconformal invariance on the string world-sheet [144,26,159]. Tentative extensions of string-theory based on extra bosonic symmetry (\mathcal{W} -symmetry) on the worldsheet have been proposed and are called \mathcal{W} -strings [53,84,287].

Symmetries of lattice models in statistical mechanics are necessarily finite-dimensional

or discrete. However, in the conformal field theory that describes their scaling limit at criticality such symmetries may give rise to continuous extensions of the conformal symmetry in the field theory. As an example we mention the \mathbb{Z}_N symmetric lattice model, $N = 2, 3, \dots$, of [113], whose scaling limit gives rise to a conformal field theory of central charge $c_N = 2(N - 1)/(N + 2)$. It has been found that this conformal field theory is invariant under the so-called \mathcal{W}_N -algebra, which is an extension of the Virasoro algebra with extra generators of spin $3, 4, \dots, N$. Another example is the XXX spin-1/2 Heisenberg spin-chain which is invariant under $A_1 = SU(2)$, and for which the associated $c = 1$ conformal field theory is invariant under the semi-direct product of the Virasoro algebra with the level-1 affine Kac-Moody algebra $A_1^{(1)}$.

In related fields which employ conformal field theory techniques, extended symmetries are equally important. In perturbed conformal field theories, the presence of \mathcal{W} -symmetries in the original conformal field theory may lead to additional integrals of motion in the perturbed theory [343]. For the relation with topological field theories, $N = 2$ extended superconformal symmetry is essential [333,334,103].

Extended symmetries appear to be particularly important for the coupling of conformal field theory ‘matter systems’ to two-dimensional gravity. It has been found [233] that there is a threshold value $c = 1$ for the central charge c of the matter system, above which the coupling of conformal matter to gravity runs into strong-coupling problems. These problems can sometimes be cured by replacing gravity by an appropriate extension, namely \mathcal{W} -gravity. Classical and quantum \mathcal{W} -gravity, in particular \mathcal{W}_3 gravity, have recently been studied by various groups (see Section 8.1).

We finally mention applications of extended conformal algebras that contain an *infinite* number of independent higher spin generators, such as w_∞ [17] and \mathcal{W}_∞ [282,283]. Some of these algebras have a clear geometrical meaning, and they play a role in field theories that are not strictly two-dimensional, such as membrane theories and the theory of self-dual $D = 4$ gravity. They have also become an important tool in the study of string theory in two-dimensional target spaces and the associated matrix models. In this context it is expected that an infinite \mathcal{W} -algebra will be part of a universal symmetry structure underlying two-dimensional string field theory.

At this point we come to the second reason for studying extended symmetries, which is the role they play in a systematic analysis of so-called rational conformal field theories.

The description of a conformal field theory that is invariant under an extension of the conformal algebra can be considerably improved by exploiting the extra symmetry. For example, degeneracies in the spectrum of conformal dimensions can be resolved by the quantum numbers that correspond to the additional symmetry. Furthermore, in rational conformal field theories the presence of extra symmetry makes it possible to have a finite decomposition of the Hilbert space of physical states in terms of irreducible representations of the extended algebra (see Section 4.1 for a discussion). If one does not extend the conformal algebra, such a finite decomposition can only be made for the minimal models of central charge $c < 1$.

Once the existence of a certain extension of the conformal algebra has been established, one can try to identify conformal field theories that realize that symmetry. Such an analysis will typically involve a study of the representation theory of the extended algebra and of the properties of their characters under modular transformations. By exploiting the requirement of modular invariance of the torus partition function (see Section 2.1), one can completely determine the possible operator contents of the conformal field theory. If the extended algebra contains fermionic currents (of half-odd-integer spin), the representations that enter the torus partition function will be subject to Gliozzi-Scherk-Olive (GSO) projections, which are familiar from the context of the superstring [176].

Some of the issues concerning the role of extended symmetries in conformal field theory have been clarified by the work on the structure theory of general rational conformal field theories, see *e.g.* [324,257,94,259]. In the systematic analysis of rational conformal field theories a central role is played by the so-called chiral algebra, which is a bosonic extension of the Virasoro algebra. There exists a precise characterization [94] of all possible rational conformal field theories with a given chiral algebra. With that result, the classification of all rational conformal field theories has formally been reduced to the study of chiral algebras and of the automorphisms of the associated fusion rules. However, since the chiral algebras of rational conformal field theories are in most cases so-called exotic or non-deformable \mathcal{W} -algebras, which appear to escape a systematic classification, the practical applicability of these results to a realistic classification program seems to be limited.

1.2. Studying extended symmetries

Some examples of extended conformal symmetries in string theory and conformal field theory have been known for a long time. Examples are the semidirect products

of the Virasoro algebra with affine Kac-Moody Lie algebras [211,256,28,171], which have for example been used to describe the propagation of strings on group manifolds [161]. Superconformal extensions go back to [266,288]; some N -extended superconformal algebras have been known since 1976 [1]. These algebras all have linear defining relations.

Shortly after the 1984 paper by Belavin, Polyakov and Zamolodchikov [29], it was realized by A.B. Zamolodchikov [342] that the extended symmetries in conformal field theory in general do not give rise to (super)algebras with *linear* defining relations. Algebras of a more general type are perfectly viable in this context. A typical feature of the more general algebras is that operator products (or, equivalently, (anti)commutators of Laurent modes) are expressed as *multilinear* expressions in the generating currents. This happens in all higher-spin bosonic extensions of the Virasoro algebra, the \mathcal{W} -algebras, but for example also in certain extended superconformal algebras [232,40,69]. Since non-linear extensions of the Virasoro algebra had not been studied in a systematic way in the mathematics literature, the challenge for physicists has been to understand these algebras and to employ them for their study of conformal field theory and string theory.

The fact that \mathcal{W} -algebras in general have non-linear defining relations puts them outside of the direct scope of Lie algebra theory. However, in recent years the structure of these algebras has been clarified to a large extent. One of the most effective tools in their study is the technique of so-called Drinfeld-Sokolov reduction, which relates \mathcal{W} -algebras to Lie algebras. This reduction also explains the fact that \mathcal{W} -algebras are related to certain hierarchies of differential equations. In fact, it was in this context that structures related to \mathcal{W} -algebras made their first appearance in the literature [155]. The simplest example of this relation is the connection of the Virasoro algebra with the second hamiltonian structure of the KdV hierarchy [243,165,163,236,15]. This then generalizes to a connection of the so-called \mathcal{W}_N algebras to the second hamiltonian structure of the generalized KdV hierarchies [2,155,339,248,16,18], and of a certain non-linear infinite \mathcal{W} -algebra, called \mathcal{W}_{KP} , to the second hamiltonian structure of the KP hierarchy [156,338,127] (see Section 5.3.4).

In order to organize our discussion in this report, we would like to distinguish the following three approaches, which have been employed in the study of extended symmetries in conformal field theory, and of the corresponding \mathcal{W} -algebras. Although they are often pursued in parallel, they are of a rather different nature.

1. The first approach is to try to write down an extended algebra by proposing a number

of extra generators (characterized by their spins which are usually chosen to be integer or halfinteger) and closing the algebra. (The form of this algebra is to a large extent fixed by the Ward identities arising from the conformal symmetry, which dictate a specific form for the operator product of two primary currents.) The difficult step is to guarantee that the proposed algebra will actually be associative (see Section 2.2). Once a consistent algebra has been found, one can try to find representations and, eventually, conformal field theory models realizing the symmetry.

2. Extended conformal algebras can be obtained in a more systematic way by employing the Drinfeld-Sokolov reduction procedure on the degrees of freedom of a theory whose structure is based on a Lie algebra (or Lie superalgebra). This approach is closely related to the study of \mathcal{W} -symmetries in Toda conformal field theories. When performed at the classical level, Drinfeld-Sokolov reduction leads to a so-called classical \mathcal{W} -algebra, which would be expressed in terms of Poisson brackets rather than commutator brackets (the two are different for non-linear algebras!). It is also possible to do the Drinfeld-Sokolov reduction at the quantum level, where it directly leads to a quantum \mathcal{W} -algebra.
3. A third approach is to take a known model of conformal field theory and to see if there are extended symmetries in that model. This means that one tries to actually construct additional currents beyond the stress-energy tensor (which corresponds to the Virasoro algebra) from the fields in the model. In the case of free fields, such constructions often turn out to be related to the Lie algebra reductions cited under 2. The most far-reaching construction of this type is the so-called coset construction, which starts from the degrees of freedom of a Wess-Zumino-Witten conformal field theory. The extended symmetries that exist in coset conformal field theories can be studied in a systematic fashion. Of course, the resulting extended algebras are associative by construction; the problem is to identify a complete set of independent generating currents.

Historically, the approach 1 was first employed by A.B. Zamolodchikov in the pioneering paper [342]. In this paper the \mathcal{W}_3 algebra was presented, and the phenomenon of ‘exotic’ \mathcal{W} -algebras (which are only consistent for some isolated values of the central charge) was first observed. In later papers, the techniques needed for direct constructions of extended conformal algebras have been refined and, with some help of computer power,

many more examples have been generated.

The approach 2, which goes back to the work of Gel'fand and Dickey [155] and of Drinfeld and Sokolov [100], was first worked out in detail by Fateev and Lykhanov [109,110,111], who in the course of their work proposed three series of \mathcal{W} -algebras based on the classical Lie algebras A_ℓ , B_ℓ and D_ℓ . In this work, the construction of these quantum \mathcal{W} -algebras was based on the quantization of the so-called Miura transformation. Later it was realized [42,92,126,118,119,137,138] that a more direct construction, called the quantum Drinfeld-Sokolov reduction, of quantum \mathcal{W} -algebras is possible.

On the level of Lagrange field theory, the theories associated with Drinfeld-Sokolov reduction are constrained Wess-Zumino-Witten theories. After implementing the constraints, these reduce to (classical or quantum) Toda field theories, where the scalar Toda fields are related to the Cartan subalgebra of the original Lie algebra. \mathcal{W} -symmetries in Toda field theories were first studied in [49,50,51,52]. The Lagrange formulation of the Drinfeld-Sokolov reduction scheme has been further worked out in [23, 24, 25].

The approach 3 to the study of \mathcal{W} -symmetries was first employed by Bais *et al.* in [11,12]. In these papers, the quantum \mathcal{W} -algebras based on the Lie algebras A_ℓ , D_ℓ and E_ℓ were independently proposed on the basis of the so-called Casimir construction for level-1 Wess-Zumino-Witten models for simply laced Lie algebras. The extension in [12] to a coset construction made it possible to obtain detailed information about the representation theory and the construction of modular invariant partition functions for these \mathcal{W} -algebras. In later work a so-called character technique was developed [62], which has made it possible to prove the existence of \mathcal{W} -symmetry in coset conformal field theories [328] and to determine the spins of the generating currents of the extended algebra.

In this report we give a comprehensive review of results obtained in the area of \mathcal{W} -symmetry in conformal field theory. We explain the basic approaches mentioned above, and discuss how these have been worked out further in recent years. Where possible we emphasize the relations that exist between different approaches.

We should stress that our selection of the material presented largely reflects our personal preferences and own contributions to the field. We have tried to avoid too much overlap with existing reviews. Our discussion in Chapters 6 and 7 is somewhat more general than the existing literature and contains original results. The style of our presentation is descriptive, reflecting a compromise between mathematical rigor and readability. Many

of the fine details are omitted; for those we refer to the original literature. We provide an extensive list of references, we hope without too many omissions. We apologize for leaving out references that would have deserved to be mentioned but that for some reason escaped our attention.

1.3. Outline of the paper

We give a brief outline on how the material in this review is organized.

We start with some preliminaries in Chapter 2. They include a brief discussion of conformal symmetry and a discussion of some technicalities concerning operator product expansions and normal ordered products. We also introduce affine Kac-Moody algebras and recall some basics of free field and Wess-Zumino-Witten conformal field theories.

In Chapter 3 we will give a definition of (quantum) \mathcal{W} -algebras. We will show the example of the \mathcal{W}_3 algebra and then present a specific class of \mathcal{W} -algebras, which are the so-called Casimir algebras for the simply laced algebras A_ℓ , D_ℓ and E_ℓ . We will also briefly introduce the notion of \mathcal{W} -superalgebras and discuss the example of the super- \mathcal{W}_3 algebra.

In Chapter 4 we will show how \mathcal{W} -algebras arise in a natural way in rational conformal field theories. We will illustrate this by giving some examples, and by briefly reviewing the structure theory of rational conformal field theories.

The main body of this paper is presented in the Chapters 5, 6 and 7, and is ordered according to the different approaches to \mathcal{W} -algebras that we listed in Section 1.2. In Chapter 5 we discuss \mathcal{W} -algebras that have been obtained through what can be called ‘direct construction’. We will discuss the method and give an extensive list of examples. In Chapter 6 we discuss the constructions based on Drinfeld-Sokolov reduction from Lie algebra valued theories. In Chapter 7 we then discuss the coset construction and show how it leads to detailed information about representation theory and modular invariants for \mathcal{W} -algebras.

In Chapter 8 we briefly discuss \mathcal{W} -gravity and we come back to applications of \mathcal{W} -symmetry in string theory. Appendix A in Chapter 9 contains our Lie algebra conventions and in Appendix B we propose a nomenclature for \mathcal{W} -algebras, trying to set a standard for later works.

2. Preliminaries

2.1. Conformal invariance: basic notions

In this section we discuss, following [29], some of the very basics of $D = 2$ conformal field theory (CFT). We will not attempt to give a self-contained account, because this alone could easily fill an issue of this journal. Some additional background can be found in Section 4.1. We refer to the literature, where excellent introductions to CFT are available [78,80,166], see also the collection of reprints [208] and references therein. Some introductory papers with particular attention for \mathcal{W} -algebras are [62,301,167,52,291,47,281,310].

The main object in a $D = 2$ CFT is the stress-energy tensor $T_{\mu\nu}(\vec{x})$, satisfying the conservation law

$$\nabla^\mu T_{\mu\nu}(\vec{x}) = 0 . \quad (2.1)$$

Because of local scale invariance it also satisfies the trace condition

$$T^\mu{}_\mu(\vec{x}) = 0 . \quad (2.2)$$

It is convenient to choose a conformal gauge $g_{\mu\nu}(\vec{x}) = \rho(\vec{x})\delta_{\mu\nu}$. (We work in a 2-dimensional Euclidean spacetime; most issues, however, easily carry over to the Minkowskian domain.)

We introduce complex coordinates $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$ (*i.e.* light-cone coordinates in a Minkowski space-time and therefore also often referred to as left and right moving coordinates, respectively). In terms of these coordinates conformal transformations are just analytic transformations $z \rightarrow w(z)$, $\bar{z} \rightarrow \bar{w}(\bar{z})$ of the coordinates z and \bar{z} . The stress energy tensor splits into two components $T \equiv T_{zz} = T_{11} - T_{22} + 2iT_{12}$, $\bar{T} \equiv T_{\bar{z}\bar{z}} = T_{11} - T_{22} - 2iT_{12}$ which, due to the conservation law (2.1), only depend on z and \bar{z} , respectively. Because both components can be treated on equal footing we will often restrict the discussion to the left moving components only.

The short-distance operator product expansion (OPE) for $T(z)$ can be argued to be

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots . \quad (2.3)$$

This relation should be understood as an identity which holds within arbitrary correlation functions. As such it is independent of the quantization scheme adopted. When using the operator formalism, where the field $T(z)$ is represented by a field-operator $T^{op}(z)$, one should keep in mind that the field product $T(z)T(w)$ in (2.3) is represented by a radially

ordered operator product, which is $T^{op}(z)T^{op}(w)$ if $|z| > |w|$ and $T^{op}(w)T^{op}(z)$ if $|z| < |w|$. Of course, this radial ordering is nothing else than the time ordering, which is familiar in the operator formalism for quantum field theories. In (2.3) the c-number c is called the central charge and the dots stand for the terms regular in the limit $z \rightarrow w$.

Among the fields in the theory there exists a preferred set which transform as tensors of weight (h, \bar{h}) under conformal transformations $z \rightarrow w(z)$, $\bar{z} \rightarrow \bar{w}(\bar{z})$

$$\phi'_{h, \bar{h}}(z, \bar{z}) = \phi_{h, \bar{h}}(w(z), \bar{w}(\bar{z})) \left(\frac{dw}{dz} \right)^h \left(\frac{d\bar{w}}{d\bar{z}} \right)^{\bar{h}} . \quad (2.4)$$

These are called primary fields of conformal dimension (h, \bar{h}) . The property that the stress energy tensor $T(z)$ is the generator of local scale transformations yields the following OPE

$$T(z)\phi_h(w) = \frac{h\phi_h(w)}{(z-w)^2} + \frac{\partial\phi_h(w)}{z-w} + \dots . \quad (2.5)$$

(The dependence on the right-moving coordinate is suppressed here. We will often do this in the sequel.) The fields in the theory which are not primary are called secondary or descendant fields. They can be obtained by taking successive operator products with $T(z)$.

In the operator formalism, it is often convenient to work with the Laurent modes L_m of the stress-energy tensor $T(z)$, which are defined by

$$T(z) = \sum_{m \in \mathbb{Z}} L_m z^{-m-2}, \quad L_m = \oint_{\mathcal{C}_0} \frac{dz}{2\pi i} z^{m+1} T(z) , \quad (2.6)$$

where the contour \mathcal{C}_0 surrounds the origin counterclockwise. The OPE relation (2.3) then translates into a commutation relation for the modes L_m

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} . \quad (2.7)$$

This is the Virasoro algebra. It is the algebra of analytic transformations of z (generated by $l_m = -z^{m+1}\frac{d}{dz}$), *i.e.* the 2-dimensional conformal group, together with a central extension. The set $\{L_{-1}, L_0, L_1\}$ generates the $sl(2, \mathbb{R})$ subalgebra of translations, global scale transformations and special conformal transformations. The vacuum $|0\rangle$ is a singlet under this subalgebra.

The OPE (2.5) for a primary field $\phi_h(z)$ translates into

$$[L_m, \phi_h(z)] = (m+1)z^m h \phi_h(z) + z^{m+1} \partial\phi_h(z) . \quad (2.8)$$

for all integers m . At this point it is useful to introduce the notion of a quasi-primary field, which is defined by the relation (2.8) for $m = -1, 0, 1$ only. This notion is thus weaker than that of a primary field. Examples of fields that are quasi-primary but not primary are

$$T(z) , \quad \Lambda(z) = (TT)(z) - \frac{3}{10}\partial^2 T(z) , \quad \dots \quad (2.9)$$

where the operator product $(TT)(z)$ is normal ordered (see Section 2.2).

Let us now say a few words about the operator product algebras of primary and quasi-primary fields. The operator product of two primary fields can be decomposed as a linear combination of other primary fields and their descendant fields

$$\begin{aligned} \phi_n(z, \bar{z})\phi_m(0, 0) = \\ \sum_p \sum_{\{k\}} \sum_{\{\bar{k}\}} C_{nm}^{p;\{k\},\{\bar{k}\}} z^{h_p-h_n-h_m+\sum_i k_i} \bar{z}^{\bar{h}_p-\bar{h}_n-\bar{h}_m+\sum_i \bar{k}_i} \phi_p^{\{k\},\{\bar{k}\}}(0, 0) . \end{aligned} \quad (2.10)$$

In this relation the index p runs over the primary fields that occur on the right hand side. The multi-indices $\{k\}$ and $\{\bar{k}\}$ label the descendants of the primary fields $\phi_p(z, \bar{z})$, which are given by

$$\phi_p^{\{k\},\{\bar{k}\}}(z, \bar{z}) = L_{-k_1} \dots L_{-k_N} \bar{L}_{-\bar{k}_1} \dots \bar{L}_{-\bar{k}_M} \phi_p(z, \bar{z}). \quad (2.11)$$

Conformal invariance implies a factorization of the constants $C_{nm}^{p;\{k\},\{\bar{k}\}}$ according to

$$C_{nm}^{p;\{k\},\{\bar{k}\}} = C_{nm}^p \beta_{nm}^{p,\{k\}} \bar{\beta}_{nm}^{p,\{\bar{k}\}}. \quad (2.12)$$

The coefficients $\beta, \bar{\beta}$ are trivial in the sense that they can be expressed in terms of the conformal dimensions h, \bar{h} , respectively. It can easily be seen that a three-point function involving fields in the conformal families of ϕ_n, ϕ_m and ϕ_p can only be nonvanishing if $C_{nm}^p \neq 0$. The coefficients C_{nm}^p can thus be viewed as three-point vertices describing the interactions in the theory. Schematically one writes this OPE as

$$[\phi_n] \cdot [\phi_m] = \sum_p C_{nm}^p [\phi_p], \quad (2.13)$$

where $[\phi_p]$ denotes the conformal family associated with the primary field $\phi_p(z, \bar{z})$.

The descendant field structure in the operator product algebra of quasi-primary fields is much simpler than it is for the primary fields: the only descendant fields to be considered

are (multiple) derivatives of the quasi-primary fields. The OPE of two chiral quasi-primary fields ϕ^i and ϕ^j of integer conformal dimensions h_i and h_j takes the general form [68]

$$\phi^i(z)\phi^j(0) = \sum_k C^{ij}_k \sum_{n=0}^{\infty} \frac{a_n^{(ijk)}}{n!} \frac{\partial^n \phi^k(0)}{z^{h_i+h_j-h_k-n}} + \gamma^{ij} \frac{1}{z^{h_i+h_j}} , \quad (2.14)$$

where k labels the quasi-primary fields occurring in the right hand side, γ^{ij} plays the role of a metric on the space of quasi-primaries and the coefficients $a_n^{(ijk)}$ are given by

$$a_n^{(ijk)} = \frac{(h_i - h_j + h_k)_n}{(2h_k)_n} , \quad (2.15)$$

with the notation $(x)_n = \Gamma(x+n)/\Gamma(x)$. In later chapters we will use the relation (2.14) as a fundamental building block in the construction of \mathcal{W} -algebras.

Let us now focus on the Hilbert space of physical states of a CFT. Conformal invariance implies that these states assemble into representations of the Virasoro algebra. The relevant representations are those for which the (left) Hamiltonian L_0 is bounded from below. They are, by convention, called highest weight modules (HWM's). The highest weight vector $|h, c\rangle$, *i.e.* the state with the lowest L_0 -eigenvalue, is characterized by the properties

$$L_0|h, c\rangle = h|h, c\rangle , \quad L_n|h, c\rangle = 0 , \quad n > 0 . \quad (2.16)$$

Before we further describe the structure of the HWM's, we mention that there exists a 1-1 correspondence between states $|\phi\rangle$ in the Hilbert space and fields $\phi(z, \bar{z})$, called vertex operators for the state $|\phi\rangle$. The correspondence is given by

$$|\phi\rangle = \lim_{z, \bar{z} \rightarrow 0} \phi(z, \bar{z})|0\rangle . \quad (2.17)$$

For a primary field $\phi_{h, \bar{h}}(z, \bar{z})$ one easily shows, using (2.8), that the associated state $|\phi\rangle$ defined by (2.17) satisfies the conditions (2.16) of a highest weight vector $|h, c\rangle_{\text{left}} \times |\bar{h}, c\rangle_{\text{right}}$. So the primary fields in the theory are in 1-1 correspondence with the highest weight vectors.

The module consisting of (finite) linear combinations of the states

$$L_{-k_1} L_{-k_2} \dots L_{-k_m} |h, c\rangle , \quad k_i > 0 , \quad (2.18)$$

is called the Verma module $M(h, c)$. The Verma module $M(h, c)$ admits an L_0 -eigenspace decomposition

$$M(h, c) = \bigoplus_{N \geq 0} M(h, c)_{(N)} , \quad (2.19)$$

where

$$M(h, c)_{(N)} = \{v \in M(h, c) | L_0 v = (h + N)v\} . \quad (2.20)$$

A basis for the eigenspace $M(h, c)_{(N)}$ is given by the states

$$L_{-k_1} \dots L_{-k_m} |h, c\rangle , \quad \sum_{i=1}^m k_i = N , \quad k_1 \geq k_2 \geq \dots k_m > 0 . \quad (2.21)$$

The dimension of the eigenspace $M(h, c)_{(N)}$ is given by ‘Euler’s partition function’ $p(N)$, *i.e.* the number of ways of partitioning N into a set of positive integers.

The hermiticity conditions

$$L_n^\dagger = L_{-n} , \quad n \in \mathbb{Z} , \quad (2.22)$$

which follow from the self-adjointness of $T(z)$, together with the normalization $\langle h, c | h, c \rangle = 1$, uniquely define a symmetric bilinear form $\langle | \rangle$ on the Verma module $M(h, c)$. It is easily seen that the eigenspace decomposition (2.19) is orthogonal with respect to this bilinear form. Let $\mathcal{M}(h, c)_{(N)}$ be the $p(N) \times p(N)$ matrix of inner products of a set of basis vectors of $M(h, c)_{(N)}$. The determinant of this matrix, which is independent of the choice of basis upto a multiplicative constant, is called the Kac determinant. It is given by [212,120]

$$\det \mathcal{M}(h, c)_{(N)} = \prod_{k=1}^N \prod_{rs=k} (h - h(r, s))^{p(N-k)} , \quad (2.23)$$

where $r, s \in \mathbb{Z}_{>0}$ and

$$h(r, s) = \frac{1}{48} \left((13 - c)(r^2 + s^2) - 24rs - 2(1 - c) + \sqrt{(1 - c)(25 - c)(r^2 - s^2)} \right) . \quad (2.24)$$

In general, the Verma module $M(h, c)$ is not irreducible, *i.e.* it contains invariant subspaces. It is easily seen that the radical of $\langle | \rangle$, consisting of the so-called null-states $v \in M(h, c)$ which are orthogonal to every state $w \in M(h, c)$, is such an invariant subspace. One can prove that $\text{Rad}(\langle | \rangle)$ is the unique maximal ideal in $M(h, c)$, implying that the coset vectorspace $L(h, c) \equiv M(h, c) / \text{Rad}(\langle | \rangle)$ is an irreducible HWM. In physical terms: states which are orthogonal to every other state decouple from all the correlation functions and can therefore be omitted altogether. Physical spectra consist therefore of irreducible HWM’s $L(h, c)$.

It is clear that states in $\text{Rad}(\langle | \rangle)$ are in 1-1 correspondence with zeros of the Kac determinant. By analyzing the Kac determinant one can therefore determine which Verma

modules are reducible (also called degenerate). In particular it is easily seen that for central charges $c > 1$ and $h > 0$ the Verma module is irreducible.

Another important issue is unitarity. A HWM $L(h, c)$ is called unitary if all the states have positive norm. The question which irreducible HWM's $L(h, c)$ are unitary can also be analyzed by means of the Kac determinant (2.23). It has been shown [146,147,148,168,169] that the requirement of unitarity restricts the possible h and c values of an irreducible HWM $L(h, c)$ to either

$$c \geq 1, \quad h \geq 0, \quad (2.25)$$

or

$$c = c(m) = 1 - \frac{6}{m(m+1)}, \quad m = 2, 3, 4, \dots, \quad (2.26)$$

in which case there are only a finite number of allowed h -values given by

$$h = h^{(m)}(r, s) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}, \quad 1 \leq r \leq m-1, \quad 1 \leq s \leq m. \quad (2.27)$$

Information on the multiplicity of states in a HWM V is contained in the character χ_V of V . This is the holomorphic function on the complex upper half plane $\{\tau \in \mathcal{C}, \text{Im}(\tau) > 0\}$, defined by

$$\chi_V(\tau) = \text{Tr}_V \left(q^{L_0 - \frac{c}{24}} \right), \quad q = e^{2\pi i \tau}. \quad (2.28)$$

From the discussion above, the character of the Verma module $M(h, c)$ is found to be

$$\chi_{M(h,c)}(\tau) = q^{h - \frac{c}{24}} \sum_{N \geq 0} p(N) q^N = \frac{q^{h - \frac{c}{24}}}{\prod_{n \geq 1} (1 - q^n)}. \quad (2.29)$$

Explicit expressions for the characters of the irreducible HWM's $L(h, c)$ can be found in [292].

Our last piece of introduction in this section concerns CFT's defined on a two-dimensional torus, which can be characterized by its modular parameter τ . It has proved interesting to study the dependence of various quantities in the CFT on τ . One such quantity is the partition function, which is formally defined as

$$Z(q) = \text{Tr} \left(q^{-\frac{c}{24} + L_0} \bar{q}^{-\frac{c}{24} + \bar{L}_0} \right), \quad (2.30)$$

where $q = e^{2\pi i \tau}$. Conformal invariance implies that the Hilbert space of the CFT splits as a sum of representations (irreducible HWM's) of the conformal algebra. Accordingly, the torus partition function has the following form

$$Z(q) = \sum_{h, \bar{h}} \chi_h(q) \mathcal{N}_{h\bar{h}} \chi_{\bar{h}}(\bar{q}), \quad (2.31)$$

where $\chi_h(q)$ is the character for the irreducible representation of the Virasoro algebra with highest weight h as in (2.28). The coefficients $\mathcal{N}_{h\bar{h}}$ are all integers and $\mathcal{N}_{00} = 1$. Modular invariance, which expresses the fact that different values for τ can give rise to the same torus, leads to the statement that the partition sum Z is invariant under the following transformations

$$T : \tau \rightarrow \tau + 1, \quad S : \tau \rightarrow -\frac{1}{\tau}. \quad (2.32)$$

The modular invariant torus partition functions for the minimal series (2.26) of central charges c have been classified in [157,76,77,223] (this is the so-called ADE classification). With this result, all unitary models of CFT with $c < 1$ are explicitly known.

At this point we stop our discussion of CFT basics. We have put some emphasis on the algebraic aspects of CFT's and on the representation theory of the Virasoro algebra, since these will be generalized by the introduction of \mathcal{W} -symmetry.

2.2. OPE's, normal ordered products and associativity

As a preparation for our discussion of \mathcal{W} -symmetry in later chapters, we will now discuss various aspects of a special subset of fields in a CFT, which are the chiral fields of integer conformal dimension, which we will call 'currents'. (Whenever needed, a 'graded' extension to a situation including chiral fields of half-integer dimension can easily be made.) For such fields the notions conformal dimension and conformal spin are the same, and we will be using both terms. It is useful to realize that the complete set of currents can be split into quasi-primary currents and currents that are (multiple) derivatives thereof.

The OPE of any two currents $A(z)$ and $B(w)$ can be written as

$$A(z)B(w) = B(w)A(z) = \sum_{r=r_0}^{\infty} \{AB\}_r(w)(z-w)^r, \quad (2.33)$$

where r runs over integer values. In general, r_0 is negative so that the OPE contains a finite number of singular terms. The singular OPE or *contraction*, which will be denoted by a hook, is given by

$$\underbrace{A(z)B(w)}_{\text{hook}} = \sum_{r<0} \{AB\}_r(w)(z-w)^r. \quad (2.34)$$

The OPE $B(z)A(w)$ is determined by (2.33) through a formal Taylor expansion.

Following [11], we define the normal ordered field product, to be denoted by $(AB)(z)$, of $A(z)$ and $B(z)$ to be the constant term in the OPE (2.33)

$$(AB)(z) \equiv \{AB\}_0(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}_z} \frac{dw}{w-z} A(w)B(z) , \quad (2.35)$$

where the contour \mathcal{C}_z encloses the point z . The normal ordered field commutator is defined as $[A, B](z) \equiv (AB)(z) - (BA)(z)$. It can be expressed in the fields occurring in the singular OPE according to

$$[A, B](z) = \sum_{r < 0} (-1)^{r+1} \frac{1}{r!} \partial^r \{AB\}_r(z) . \quad (2.36)$$

There exists a simple calculus, which allows one to compute with contractions and normal ordered products of composite fields. In particular, there is the Wick theorem for the contraction of $A(z)$ with the composite field $(BC)(w)$

$$\underbrace{A(z)}{\text{contract}}(BC)(w) = \frac{1}{2\pi i} \oint_{\mathcal{C}_w} \frac{dx}{x-w} \{ \underbrace{A(z)}{\text{contract}} B(x) C(w) + B(x) \underbrace{A(z)}{\text{contract}} C(w) \} . \quad (2.37)$$

We remark that the normal ordered product as defined in (2.35) is neither commutative (this we saw in (2.36)) nor associative. Using

$$(A(BC))(z) - (B(AC))(z) = ([A, B]C)(z) \quad (2.38)$$

one can compute the associator $(A(BC))(z) - ((AB)C)(z)$ and find it to be nonvanishing in general. (Let us stress that this fact is not at all in conflict with the property that the full operator product algebra (OPA) is required to be associative.) More complicated rearrangement lemmas, for example for the difference between $((AB)(CD))(z)$ and $(A(B(CD)))(z)$, have been given in appendix A of [11]. The rules for calculating operator product expansions and rearranging normal ordered expressions have been implemented in a Mathematica package [318].

We define the modes A_m of a current $A(z)$ of conformal dimension h_A by

$$A_m = \frac{1}{2\pi i} \oint_{\mathcal{C}_0} dz A(z) z^{m+h_A-1}, \quad A(z) = \sum_{m \in \mathbb{Z}} A_m z^{-m-h_A} \quad (2.39)$$

The contraction $A(z)B(w)$ can then be translated into a commutation relation $[A_m, B_n]$ for the modes A_m and B_n . In particular, the OPE (2.5) translates into

$$[L_m, A_n] = ((h_A - 1)m - n)A_{m+n} . \quad (2.40)$$

An important property is the *associativity* of the commutator algebra of the modes of all currents in a CFT. It is expressed by the Jacobi identity,

$$[A_m, [B_n, C_r]] + [C_r, [A_m, B_n]] + [B_n, [C_r, A_m]] = 0. \quad (2.41)$$

At the level of currents, the property of associativity is most easily expressed by considering four-point correlation functions. If one denotes the currents by $A^I = A^i(z_i)$ and formally writes the OPE's as $A^I A^J = \sum_K C^{IJ}_K A^K$, then the associativity condition can be written as

$$\sum_M C^{IJ}_M C^{MK}_L = \sum_M C^{IM}_L C^{JK}_M \quad (2.42)$$

As explained in App B. of [29], this property precisely corresponds to the so-called crossing symmetry of the 4-point functions involving the currents A^I , A^J , A^K and A^L . Crossing symmetry simply means that the four-point function $\langle A^i(z_i) A^j(z_j) A^k(z_k) A^l(z_l) \rangle$ is invariant under all permutations of the labels i, j, k and l . We can easily work out this condition in the case where all four currents are quasi-primary: using the $sl(2, \mathbb{R})$ invariance, we can then write

$$\begin{aligned} & \langle A^i(z_i) A^j(z_j) A^k(z_k) A^l(z_l) \rangle = \\ & G \begin{pmatrix} j & i \\ k & l \end{pmatrix} (x) \frac{(z_j - z_l)^{-h_j - h_k - h_l + h_i} (z_i - z_j)^{-h_i - h_j + h_k + h_l}}{(z_i - z_k)^{2h_k} (z_i - z_l)^{-h_i - h_l + h_j + h_k}}, \end{aligned} \quad (2.43)$$

where

$$G \begin{pmatrix} j & i \\ k & l \end{pmatrix} (x) = \langle i | A^j(1) A^k(x) | l \rangle, \quad x = \frac{(z_i - z_j)(z_k - z_l)}{(z_i - z_k)(z_j - z_l)}, \quad (2.44)$$

and we defined (compare with (2.17))

$$|i\rangle = \lim_{z_i \rightarrow 0} A^i(z_i) |0\rangle, \quad \langle i| = \lim_{z_i \rightarrow 0} z_i^{-2h_i} \langle 0 | A^i(1/z_i). \quad (2.45)$$

[The expressions (2.44) and (2.45) refer to the operator formalism and define the 4-point-function for $|x| < 1$. The result for $|x| \geq 1$ is obtained by analytic continuation.] The crossing symmetry conditions that correspond to a change of channels are given by

$$\begin{aligned} G \begin{pmatrix} j & i \\ k & l \end{pmatrix} (x) &= (-1)^{h_i + h_j + h_k + h_l} x^{-2h_k} G \begin{pmatrix} j & l \\ k & i \end{pmatrix} \left(\frac{1}{x}\right) \\ G \begin{pmatrix} j & i \\ k & l \end{pmatrix} (x) &= (-1)^{h_i + h_j + h_k + h_l} G \begin{pmatrix} l & i \\ k & j \end{pmatrix} (1-x). \end{aligned} \quad (2.46)$$

The connection between the characterization of associativity via the Jacobi identity for modes and via the condition for crossing symmetry of four-point functions of currents

has been discussed in [68]. The two are believed to be equivalent, but a direct proof of this has (to our knowledge) not been given.

There exists a third way to characterize associativity, which was first discussed in [61]. It uses the following property of the normal ordered field commutators which we introduced above

$$[A, [B, C]](z) + [C, [A, B]](z) + [B, [C, A]](z) = 0. \quad (2.47)$$

(This property holds despite the fact that the normal ordered product by itself is not associative!) In many examples [61,4] it has been found that this condition, which is certainly a necessary condition for associativity of the OPA, allows one to quickly derive associativity constraints on the central charge of a proposed extended algebra. We expect that the condition, when imposed for all currents in the algebra, is also a sufficient condition for associativity, but this has not been proven.

2.3. Auxiliary field theories

2.3.1. Free fields

In this section we briefly discuss the conformal field theory of a single free massless scalar field $\phi(z, \bar{z})$ and the Feigin-Fuchs construction. We shall also indicate two alternative free field constructions for the affine Kac-Moody algebra $A_1^{(1)}$, both of which can be generalized to other algebras. (See Section 2.3.2 for the definition of affine Kac-Moody algebras.)

We first consider a single free scalar field $\phi(z, \bar{z})$ which is compactified on a circle $\mathbb{R}/2\pi R$ of radius R . The action of this field theory is given by

$$S[\phi] = -\frac{1}{2} \int d^2z \partial^\alpha \phi \partial_\alpha \phi \quad (2.48)$$

and the corresponding equation of motion is the 2-dimensional Laplace equation. The most general solution is given by

$$\phi(z, \bar{z}) = \varphi(z) + \bar{\varphi}(\bar{z}), \quad (2.49)$$

where φ and $\bar{\varphi}$ are single-valued functions on the complex plane. In terms of a mode expansion we have

$$i\partial\varphi = \sum_{n \in \mathbb{Z}} \alpha_n z^{-n-1}, \quad (2.50)$$

and an analogous expression for $\overline{\partial\varphi}(\bar{z})$. Canonical quantization gives the following commutation relations

$$[\alpha_m, \alpha_n] = m \delta_{m+n,0} , \quad (2.51)$$

which are equivalent to the contraction

$$\underbrace{\partial\varphi(z)\partial\varphi(w)} = \frac{-1}{(z-w)^2} . \quad (2.52)$$

This contraction, or equivalently the commutation relations (2.51), defines the $U(1)$ affine Kac-Moody algebra.

The stress energy tensor is given by

$$T(z) = -\frac{1}{2}(\partial\varphi\partial\varphi)(z) , \quad (2.53)$$

with the parentheses denoting normal ordering as in Section 2.2. In terms of modes we have in particular

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n \geq 1} \alpha_{-n}\alpha_n . \quad (2.54)$$

By invoking the Wick theorem, we can compute the OPA of $T(z)$, which is precisely given by the result (2.3) with $c = 1$.

The spectrum of the scalar field model is constructed as a Verma module over a highest weight vector $|\lambda, \bar{\lambda}\rangle$. Explicitly

$$\begin{aligned} \alpha_0|\lambda, \bar{\lambda}\rangle &= \lambda|\lambda, \bar{\lambda}\rangle , & \bar{\alpha}_0|\lambda, \bar{\lambda}\rangle &= \bar{\lambda}|\lambda, \bar{\lambda}\rangle \\ \alpha_n|\lambda, \bar{\lambda}\rangle &= \bar{\alpha}_n|\lambda, \bar{\lambda}\rangle = 0 , & n &> 0 , \end{aligned} \quad (2.55)$$

and a basis is given by the states

$$\begin{aligned} \alpha_{-k_1} \dots \alpha_{-k_m} \bar{\alpha}_{-l_1} \dots \bar{\alpha}_{-l_n} |\lambda, \bar{\lambda}\rangle , & \quad k_1 \geq k_2 \geq \dots > 0 , \\ & \quad l_1 \geq l_2 \geq \dots > 0 . \end{aligned} \quad (2.56)$$

From (2.54) we find that the highest weight vector $|\lambda, \bar{\lambda}\rangle$ has conformal dimension $(h, \bar{h}) = (\frac{1}{2}\lambda^2, \frac{1}{2}\bar{\lambda}^2)$.

To determine the set $(\lambda, \bar{\lambda})$ that occur in the theory one observes that, by using (2.17), the highest weight vectors $|\lambda, \bar{\lambda}\rangle$ are obtained, from the vertex operators

$$V_{\lambda, \bar{\lambda}}(z, \bar{z}) = V_\lambda(z)V_{\bar{\lambda}}(\bar{z}) , \quad (2.57)$$

where

$$V_\lambda(z) = (e^{i\lambda\varphi})(z) , \quad V_{\bar{\lambda}}(\bar{z}) = (e^{i\bar{\lambda}\bar{\varphi}})(\bar{z}) . \quad (2.58)$$

Locality requires that the spin $h - \bar{h}$ is an integer. Together with the invariance under $\phi \rightarrow \phi + 2\pi R$ this restricts the momenta $(\lambda, \bar{\lambda})$ to be on the lattice

$$\Gamma = \left\{ (\lambda, \bar{\lambda}) = \left(\frac{n}{R} + \frac{1}{2}mR, \frac{n}{R} - \frac{1}{2}mR \right) \mid n, m \in \mathbb{Z} \right\} . \quad (2.59)$$

The partition function is now easily computed

$$Z = \frac{\sum_{(\lambda, \bar{\lambda}) \in \Gamma} q^{\frac{1}{2}\lambda^2} \bar{q}^{\frac{1}{2}\bar{\lambda}^2}}{\left| q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n) \right|^2} . \quad (2.60)$$

A special situation occurs for $R = \sqrt{2}$, where there are three fields of dimension $(1, 0)$ namely

$$J^-(z) = \left(e^{-i\sqrt{2}\varphi} \right) (z) , \quad J^0(z) = i\partial\varphi(z) , \quad \text{and} \quad J^+(z) = \left(e^{i\sqrt{2}\varphi} \right) (z) . \quad (2.61)$$

The OPA of these three currents closes; this is the so-called vertex operator construction for the affine Kac-Moody algebra $A_1^{(1)}$ at level 1 [180,27].

Let us now briefly discuss a small modification of the scalar field model, known as the Feigin-Fuchs construction [120]. This construction is most useful in the study of minimal models, both of the Virasoro algebra, and, as we shall see later, of minimal models of \mathcal{W} -algebras. The modification consists of putting a background charge Q at $z = \infty$. This does not affect the OPE (2.52), but it does modify the stress energy tensor to

$$T(z) = -\frac{1}{2}(\partial\varphi\partial\varphi)(z) + iQ\partial^2\varphi(z) . \quad (2.62)$$

The modified stress energy tensor (2.62) again satisfies the OPE (2.3), but now with a central charge

$$c = 1 - 12Q^2 . \quad (2.63)$$

The effect of the background charge is thus to screen the central charge of the original system. Another consequence of the background charge is that the vertex operators

$$V_\lambda(z) = (e^{i\lambda\varphi})(z) , \quad (2.64)$$

now become primary fields of conformal dimension

$$h(\lambda) = \frac{1}{2}\lambda(\lambda - 2Q) . \quad (2.65)$$

Using this modification it is possible to give an explicit construction of null-states, *i.e.* the zero's of the Kac determinant, and thus to prove (2.23). The application of the Feigin-Fuchs construction to minimal models has been further worked out by Felder in [124]. In our discussion of \mathcal{W} -algebras in Chapters 6 and 7, we will be using multi-component extensions of the Feigin-Fuchs construction.

In order to define an alternative free field realization of $A_1^{(1)}$, we introduce bosonic first order fields $\beta(z)$ and $\gamma(z)$ through the contraction

$$\underbrace{\gamma(z)\beta(w)} = \frac{1}{z-w} . \quad (2.66)$$

One easily checks that the currents

$$\begin{aligned} J^-(z) &= -(\gamma\gamma\beta)(z) - \sqrt{2(k+2)}(\gamma i\partial\varphi)(z) - k\partial\gamma(z) \\ J^0(z) &= 2(\gamma\beta)(z) + \sqrt{2(k+2)}i\partial\varphi(z) \\ J^+(z) &= \beta(z) \end{aligned} \quad (2.67)$$

satisfy the OPA of the level k affine Kac-Moody algebra $A_1^{(1)}$ [325]. The similar construction for more general Lie algebras [117,64,65] will be used in Chapter 6.

2.3.2. Affine Kac-Moody algebras and WZW models

The theory of affine Kac-Moody algebras (or AKM algebras in brief) has played a crucial role in the mathematical analysis of various CFT's and \mathcal{W} -algebras. It is actually possible to use representation spaces of AKM algebras to construct a Hilbert space for a conformal field theory. The symmetry currents in the CFT (such as the stress energy tensor) are then constructed from currents taking values in the underlying finite-dimensional Lie algebra. Examples for this are the Sugawara and coset constructions and generalizations that one obtains by solving the Virasoro master equation [181]. In the unitary case, these constructions all start from integrable representations of AKM algebras, which occur at integer level k . In the quantum Drinfeld-Sokolov reduction scheme, which we discuss in Chapter 6, representations of AKM algebras of fractional level form the starting point.

Before we say more about the applications, we introduce current fields $J(z)$ and $\bar{J}(\bar{z})$ (satisfying $\partial_{\bar{z}}J = \partial_z\bar{J} = 0$), that take values in a finite-dimensional Lie algebra \mathfrak{g} . (We will mostly restrict ourselves to the left moving components $J(z)$ only.) Choosing a set of anti-hermitean generators $\{T_a, a = 1, \dots, \dim(\mathfrak{g})\}$ ¹,

$$[T_a, T_b] = f_{ab}{}^c T_c, \quad (2.68)$$

the components $J^a(z)$ satisfy the OPE

$$J^a(z)J^b(w) = \frac{k d^{ab}}{(z-w)^2} + f_{ab}{}^c \frac{J^c(w)}{z-w} + \dots \quad (2.69)$$

The Cartan-Killing metric

$$d_{ab} = \text{Tr}(T_a T_b) \quad (2.70)$$

is used to raise and lower indices.² The c-number k is called the central charge or level. The following commutation relations for the Laurent modes J_m^a , $m \in \mathbb{Z}$, of $J^a(z) = \sum_{m \in \mathbb{Z}} J_m^a z^{-m-1}$ are equivalent to the OPE (2.69)

$$[J_m^a, J_n^b] = f_{ab}{}^c J_{m+n}^c + km d^{ab} \delta_{m+n,0}. \quad (2.71)$$

The algebra (2.71), which is denoted by $\widehat{\mathfrak{g}}$ or $\mathfrak{g}^{(1)}$, is called an (untwisted) affine Kac-Moody (AKM) algebra, or affine Lie algebra in brief. In the mathematics literature these algebras were first discussed in [211,256], in the physics literature they made their first appearance in [28]; see [214,216] for their mathematical aspects and [171] for an introduction from a physicist's point of view. Notice that the zero modes J_0^a satisfy the commutation relations of the underlying finite dimensional Lie algebra \mathfrak{g} .

The most interesting representations of an AKM algebra (2.71) are the irreducible highest weight modules (HWM's), which are characterized by a highest weight $\hat{\Lambda}$. The projection Λ of the weight $\hat{\Lambda}$ onto the weight lattice of the underlying Lie algebra \mathfrak{g} characterizes the \mathfrak{g} -representation of the highest weight state. A special class of these highest weight representations, which occur for integer level k , are the so-called integrable HWM's for which Λ is an integral dominant weight. For given integer level k only a finite

¹ In later chapters we will mainly use the Chevalley basis $\{e^\alpha, e^{-\alpha}, h^i\}$.

² Our normalizations are chosen such that $|\alpha|^2 = 2$ for a long root α of \mathfrak{g} . The trace is taken over the fundamental representation of \mathfrak{g} .

number of integrable HWM's $L(\Lambda)$ exist. For fractional levels the more general notion of 'modular invariant representations' was introduced in [217].

The most direct application of AKM current algebra to CFT is through the so-called Sugawara construction, which gives the following expression for a CFT stress-energy tensor $T(z)$ in terms of AKM currents $J^a(z)$

$$T(z) = \frac{1}{2(k + h^\vee)} d_{ab} (J^a J^b)(z) , \quad (2.72)$$

The c-number h^\vee is the dual Coxeter number of \mathfrak{g} [214]. Using (2.69) one can verify that $T(z)$ satisfies the Virasoro OPE (2.3), with central charge

$$c = c(\widehat{\mathfrak{g}}, k) = \frac{k(\dim \mathfrak{g})}{k + h^\vee} . \quad (2.73)$$

To each highest weight Λ of a HWM $L(\Lambda)$ of the AKM algebra one can now associate a Virasoro primary field ϕ_Λ , which has conformal dimension

$$h_\Lambda = \frac{c_\Lambda}{2(k + h^\vee)} , \quad (2.74)$$

where $c_\Lambda = (\Lambda, \Lambda + 2\rho)$ is the eigenvalue of the second order Casimir in the \mathfrak{g} -representation characterized by the highest weight Λ .

The CFT's that correspond to the form (2.72) of the stress-energy tensor are the so-called Wess-Zumino-Witten (WZW) conformal field theories [332,331]. A WZW model is a nonlinear sigma model on a compact group manifold G , which contains a topological term, the Wess-Zumino term [331], in its action. The presence of this Wess-Zumino term requires the level k to be an integer. The model is conformally invariant provided the relative coefficient of the Wess-Zumino term is chosen in a specific way [332]. The chiral stress-energy tensor precisely takes the form (2.72), where \mathfrak{g} is now the Lie algebra of the group G .

The spectrum of the WZW models was determined by Gepner and Witten [161], who found that only integrable HWM's can occur in the spectrum. Knizhnik and Zamolodchikov [234] showed how to apply the techniques of [29] to the WZW model, and derived differential equations for the correlation functions. These so-called Knizhnik-Zamolodchikov equations in principle solve the WZW model.

WZW models defined on a two-dimensional torus are characterized by their modular invariant partition function. For the case when the underlying Lie algebra is $A_1^{(1)}$ all

possible modular invariants have been classified in [157,76,77,223]. For higher rank algebras such complete results have, to our knowledge, not been obtained.

One can thus construct HWM's of the Virasoro algebra from HWM's of AKM algebras by means of the Sugawara construction. Clearly, the scope of this construction is limited since it assumes the presence of an affine Kac-Moody symmetry in the CFT. There exist, however, more general Virasoro constructions. One of those is the so-called Goddard-Kent-Olive coset construction [168,169], which associates a Virasoro algebra to a coset pair $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}')$, $\widehat{\mathfrak{g}}' \subset \widehat{\mathfrak{g}}$ of AKM algebras³. Examples of the coset construction were already given in [28,179]. The coset construction starts from the Sugawara tensors $T(z)$ and $T'(z)$ associated to $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{g}}'$, which generate Virasoro algebras of central charges $c(\widehat{\mathfrak{g}}, k)$ and $c(\widehat{\mathfrak{g}}', k')$, respectively. One then constructs the operator

$$\widetilde{T}(z) = T(z) - T'(z) , \quad (2.75)$$

which generates a so-called 'coset' Virasoro algebra, of central charge $c(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', k)$ given by⁴

$$c(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', k) = c(\widehat{\mathfrak{g}}, k) - c(\widehat{\mathfrak{g}}', k') . \quad (2.76)$$

An important property of this coset Virasoro algebra is that it commutes with the AKM subalgebra $\widehat{\mathfrak{g}}'$. This implies that the $\widehat{\mathfrak{g}}$ HWM's can naturally be interpreted as HWM's of the direct sum of $\widehat{\mathfrak{g}}'$ and the coset Virasoro algebra. In particular, the coset vector space obtained from a $\widehat{\mathfrak{g}}$ HWM by identifying states in the same $\widehat{\mathfrak{g}}'$ HWM, is a (not necessarily irreducible) HWM of the coset Virasoro algebra.

The coset construction makes it possible to relate modular invariant partition functions for the AKM algebra $A_1^{(1)}$ and for the Virasoro algebra [60,70].

General Virasoro constructions that are quadratic in the currents of an AKM $\widehat{\mathfrak{g}}$ have been studied in a systematic fashion (see [181] for a recent review and references to the original literature). These constructions correspond to solutions of the so-called Virasoro master equation. Some of the solutions that have been found have irrational Virasoro

³ The central charge k' of the AKM subalgebra $\widehat{\mathfrak{g}}'$ is determined by $k' = jk$, where j is the Dynkin index of the embedding $\widehat{\mathfrak{g}}' \subset \widehat{\mathfrak{g}}$.

⁴ A special situation arises when $c(\widehat{\mathfrak{g}}, k) = c(\widehat{\mathfrak{g}}', k')$, in which case the embedding $\widehat{\mathfrak{g}}' \subset \widehat{\mathfrak{g}}$ is called a *conformal* embedding. For such embeddings, which were classified in [297,10], the coset Virasoro algebra is trivial.

central charge and are thus non-rational CFT's (compare with Section 4.1). The study of the Virasoro master equation has revealed an interesting connection with the theory of graphs and generalized graphs.

In later sections we will argue that the Sugawara and coset constructions as discussed here are not complete if the Lie algebra \mathfrak{g} involved has rank greater than 1, *i.e.* if it is not A_1 . The extension of the Sugawara construction to the so-called Casimir construction, to be discussed in Section 3.2, will naturally lead to a class of \mathcal{W} -algebras, the so-called Casimir algebras. A similar extension of the coset-construction, to be discussed in Chapter 7, will allow us to construct unitary HWM's and modular invariant partition functions for these \mathcal{W} -algebras. More general constructions of \mathcal{W} -algebras from the currents of AKM algebras have also been considered [88].

3. \mathcal{W} -algebras and Casimir algebras

3.1. \mathcal{W} -algebras: definitions and the example of \mathcal{W}_3

In the previous chapter we discussed some basics of conformal symmetry and its elementary realizations in terms of free fields and of the currents of affine Kac-Moody algebras. We will now focus on the central topic of this paper: \mathcal{W} -algebras and related structures.

Our first concern is to make more precise what we mean by a \mathcal{W} -algebra. Since our main interest is in the quantum \mathcal{W} -algebras that occur in CFT's, we will give a definition, following [167], that is tailored for this purpose. However, we would like to stress that the mathematical notions of both classical and quantum \mathcal{W} -algebras are purely algebraic concepts which exist independently from the specific context of CFT.

Following [167], we shall first define the notion of a meromorphic conformal field theory. A quantum \mathcal{W} -algebra can then be defined to be a meromorphic conformal field theory of a special type.

A meromorphic conformal field theory (mcft) consists of a “characteristic Hilbert space” \mathcal{H} (we will sometimes refer to \mathcal{H} as the “vacuum module”) and a map $|\psi\rangle \rightarrow V(|\psi\rangle, z)$, the so-called “vertex operator map,” from \mathcal{H} into the space of fields.

Furthermore, there should be a distinguished state $|L\rangle$, whose corresponding vertex operator $T(z) = V(|L\rangle, z)$ is the stress-energy tensor of the theory. Its modes $T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ satisfy the Virasoro algebra.

The vertex operator map has to satisfy the following properties:

- (i) There exists a unique state $|0\rangle \in \mathcal{H}$ such that $V(|\psi\rangle, z)|0\rangle = e^{zL-1}|\psi\rangle$
- (ii) $\langle\psi_1|V(|\psi\rangle, z)|\psi_2\rangle$ is a meromorphic function of z
- (iii) $\langle\psi_1|V(|\psi\rangle, z)V(|\chi\rangle, w)|\psi_2\rangle$ is a meromorphic function for $|z| > |w|$
- (iv) $\langle\psi_1|V(|\psi\rangle, z)V(|\chi\rangle, w)|\psi_2\rangle = \epsilon_{\psi\chi}\langle\psi_1|V(|\chi\rangle, w)V(|\psi\rangle, z)|\psi_2\rangle$ by analytic continuation.

Here $\epsilon_{\psi\chi} = -1$ if both ψ and χ are fermionic, and 1 otherwise.

From these axioms it follows that the vertex operator map $|\psi\rangle \rightarrow V(|\psi\rangle, z)$ is in fact an isomorphism.

Let $\mathcal{H} = \bigoplus_h \mathcal{H}_h$ denote the decomposition of \mathcal{H} into L_0 eigenspaces of eigenvalue h (“conformal dimensions”). The above axioms imply that only integer or half-odd-integer conformal dimensions h can occur in a mcft. The fields of integer dimension are bosonic

and those of half-odd-integer dimension are fermionic. When $|\psi\rangle \in \mathcal{H}_h$, we will use the following mode expansion for the corresponding field (compare with (2.39))

$$V(|\psi\rangle, z) = \sum_n \psi_n z^{-n-h} \quad (3.1)$$

where $n \in \mathbb{Z}$ for h integer and $n \in \mathbb{Z} + \frac{1}{2}$ for h half-odd integer. It follows that

$$\begin{aligned} \psi_{-h}|0\rangle &= |\psi\rangle \\ \psi_n|0\rangle &= 0 \quad \text{for } n \geq -h + 1 \end{aligned} \quad (3.2)$$

Moreover, the operator product expansion of two fields $A(z)$ and $B(w)$ of conformal dimensions h_A and h_B may be shown to equal (2.33), where

$$\{AB\}_r(w) = V(A_{-r-h_A} B_{-h_B}|0\rangle, w) \quad (3.3)$$

In particular, the normal ordered product $(AB)(z)$ corresponds to the state $A_{-h_A} B_{-h_B}|0\rangle$.

A quantum \mathcal{W} -algebra can now be defined to be a meromorphic conformal field theory of a special type. This will directly imply that the \mathcal{W} -algebra satisfies a number of CFT consistency conditions; in particular, it guarantees that the operator product algebra of the \mathcal{W} -currents will be associative.

Definition

A quantum \mathcal{W} -algebra is a meromorphic conformal field theory whose characteristic Hilbert space \mathcal{H} contains a finite number of distinguished states $|i\rangle$, including the state $|L\rangle$, whose corresponding vertex operators $W^{(s_i)}(z) = V(|i\rangle, z)$ ($T(z) = W^{(2)}(z)$) are quasiprimary fields of integer conformal dimension s_i . Furthermore, it is required that the entire space of fields is spanned by normal ordered products of the fields $W^{(s_i)}(z)$ and their derivatives.

It can be shown from the definition that \mathcal{H} is spanned by lexicographically ordered states

$$W_{-m_1-s_{i_1}}^{(s_{i_1})} \dots W_{-m_n-s_{i_n}}^{(s_{i_n})} |0\rangle \quad (3.4)$$

where $s_{i_j} \geq s_{i_{j+1}}$, $i_j = i_{j+1} \Rightarrow m_j \geq m_{j+1}$ and $m_j \geq 0$. Conversely, if \mathcal{H} is spanned by states of this form then all the fields can be written as normal ordered products of the $W^{(s_i)}(z)$ and their derivatives.

Remarks

1. *Generalizations.* It is of course possible to relax this definition in various directions. If we allow an infinite set of extra currents, we can include important examples such as \mathcal{W}_∞ (see Section 5.2.1). Also, one can easily define a graded version of \mathcal{W} -algebras by allowing the spins of the generating currents to become half-integer (see Section 3.3 for an example).

2. *Modes.* As we discussed in Chapter 2, an OPE algebra of currents is equivalent to the commutator algebra of their Laurent modes, which will be $L_m, (TT)_m, \dots, W_m^{(s_i)}, \dots$ in the case of a \mathcal{W} -algebra. The Jacobi identity for this commutator algebra is equivalent to the condition of crossing symmetry for four-point functions of currents.

In some cases it is possible to assign half-integer (or, in one case, 1/3-integer) modes to some of the currents in a \mathcal{W} -algebra [184,185,186]. The corresponding twisted sectors of the algebra can be compared to the Ramond sectors of the superconformal algebras. The twisting changes the structure of the representation theory and leads to new modular invariant partition functions. In the Chapters 5 and 6 we will say more about twisted \mathcal{W} -algebras.

It is thus possible to carry out the analysis of \mathcal{W} -algebras entirely on the level of modes. In this report we will mostly use the formulation in terms of currents instead.

3. *Generic vs exotic algebras.* The OPE T - T (see (2.3)) contains the real parameter c , called the central charge. In certain cases it turns out to be possible to define a one-parameter family of consistent \mathcal{W} -algebras, the parameter being c . In such cases various properties of the algebra, and the representation theory, can be studied as a function of c . We will refer to these algebras by saying that they are of *generic* type. In contrast to this there is the case where a \mathcal{W} -algebra with a certain set of currents only exists for some isolated values of the central charge c . We will denote the latter type as *exotic*. These notions will become more clear when we discuss examples of both possibilities in later sections.⁵

⁵ In a recent paper [72], the terminology ‘deformable’ and ‘non-deformable’ is proposed for what we call ‘generic’ and ‘exotic’ \mathcal{W} -algebras, respectively. As further pieces of terminology, the notions of ‘positive-definiteness’ and ‘reductivity’ of a \mathcal{W} -algebra are introduced in [72].

The prototype example of all \mathcal{W} -algebras is the \mathcal{W}_3 algebra⁶ introduced by A.B. Zamolodchikov in [342]. It has generators $T(z)$ and $W(z)$, where $W(z)$ is primary of spin 3 with respect to $T(z)$. The singular OPE of the spin-3 currents reads

$$\begin{aligned} \underbrace{W(z)W(w)} &= \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\ &+ \frac{1}{(z-w)^2} \left[2\beta\Lambda(w) + \frac{3}{10}\partial^2 T(w) \right] \\ &+ \frac{1}{(z-w)} \left[\beta\partial\Lambda(w) + \frac{1}{15}\partial^3 T(w) \right], \end{aligned} \quad (3.5)$$

where

$$\Lambda(w) = (TT)(w) - \frac{3}{10}\partial^2 T(w), \quad (3.6)$$

and β is given by

$$\beta = \frac{16}{22 + 5c}. \quad (3.7)$$

The central charge parameter c is arbitrary; in the terminology of ‘generic’ vs. ‘exotic’ \mathcal{W} -algebras the \mathcal{W}_3 algebra is thus an algebra of generic type. Zamolodchikov showed that these OPE’s are the most general compatible with the requirement of crossing symmetry. The four-point function of the currents $W(z)$ is given by (compare with (2.43), (2.44))

$$\langle W(z_i)W(z_j)W(z_k)W(z_l) \rangle = G(x) (z_i - z_k)^{-6} (z_j - z_l)^{-6} \quad (3.8)$$

where

$$\begin{aligned} G(x) &= \frac{c^2}{9} \left[\frac{1}{x^6} + \frac{1}{(1-x)^6} + 1 \right] \\ &+ 2c \left[\frac{1}{x^4} + \frac{1}{(1-x)^4} + \frac{1}{x^3} + \frac{1}{(1-x)^3} - \frac{1}{x} - \frac{1}{(1-x)} \right] \\ &+ c \left(\frac{9}{5} + \frac{32}{5} \frac{16}{(22+5c)} \right) \left[\frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{2}{x} + \frac{2}{(1-x)} \right] \end{aligned} \quad (3.9)$$

with x as in (2.44). It can easily be checked that the conditions (2.46) for crossing symmetry are indeed satisfied.

For completeness, we now give the commutator algebra of the Laurent modes L_n and W_n , which are defined as in (2.39). The commutators $[L_m, L_n]$ and $[L_m, W_n]$ are as in

⁶ For a more systematic nomenclature for \mathcal{W} -algebras, we refer to later chapters and to Appendix B.

(2.7) and (2.40) (with $h_W = 3$) and we have

$$\begin{aligned}
[W_m, W_n] &= \frac{c}{360} m(m^2 - 1)(m^2 - 4)\delta_{m+n,0} \\
&+ (m - n) \left\{ \frac{1}{15}(m + n + 3)(m + n + 2) - \frac{1}{6}(m + 2)(n + 2) \right\} L_{m+n} \\
&+ \beta(m - n)\Lambda_{m+n},
\end{aligned} \tag{3.10}$$

where

$$\Lambda_m = \sum_n (L_{m-n}L_n) - \frac{3}{10}(m + 3)(m + 2)L_m. \tag{3.11}$$

Later in this review we will come back to the \mathcal{W}_3 algebra a number of times.

With the example of \mathcal{W}_3 at hand, we would like to make a few further remarks on the general structure of \mathcal{W} -algebras. Our definition makes it clear that the regular terms in any OPE do not contain any independent information, since they can easily be expressed in terms of the original currents. However, we would like to stress that the entire set of singular and regular terms in any OPE forms a representation of conformal symmetry (*i.e.* the Virasoro algebra), which is in general infinitely reducible. Concretely, this implies that the set of all terms in the OPE of two primary fields can be split as a sum of primary fields and descendant fields that are related to primary fields that appeared in more singular terms in the expansion (as in (2.10)). Thus, in general OPE's, an infinite number of primary fields of the Virasoro algebra are present.

Let us illustrate this with the OPE W - W in the \mathcal{W}_3 algebra. We consider the first regular term in this OPE, which is simply given by $(WW)(w)$. Clearly, this term is not just a Virasoro descendant of the identity. Instead, we can 'split' it in terms of a descendant field, given by

$$\begin{aligned}
&\alpha (T\partial^2 T - \partial(T\partial T)) (z) + \gamma (T(TT)) (z) + \delta \partial^4 T(z) \\
&+ \epsilon \partial \left(\frac{1}{15} \partial^3 T + \beta \partial \left(T^2 - \frac{3}{10} \partial^2 T \right) \right) (z)
\end{aligned} \tag{3.12}$$

plus a new Virasoro primary field $\Phi^{(6)}(z)$, which is given by

$$\begin{aligned}
\Phi^{(6)}(z) &= (WW)(z) - \alpha (T\partial^2 T - \partial(T\partial T)) (z) - \gamma (T(TT)) (z) \\
&- \delta \partial^4 T(z) - \epsilon \partial \left(\frac{1}{15} \partial^3 T + \beta \partial \left(T^2 - \frac{3}{10} \partial^2 T \right) \right) (z).
\end{aligned} \tag{3.13}$$

In here, α , γ , δ and ϵ are functions of c given by

$$\begin{aligned}\alpha &= \frac{43c^2 - 1266c - 3784}{2(22 + 5c)(68 + 7c)(2c - 1)}, & \gamma &= \frac{16(191c + 22)}{3(22 + 5c)(68 + 7c)(2c - 1)}, \\ \delta &= \frac{-189c^3 + 812c^2 - 18900c - 12320}{336(22 + 5c)(68 + 7c)(2c - 1)}, & \epsilon &= \frac{67c^2 + 178c - 752}{16(68 + 7c)(2c - 1)}\end{aligned}\quad (3.14)$$

and β is as in (3.7). We mention here that $\Phi^{(6)}(z)$, which is primary for all values of c , is actually a null field if we choose c to be $4/5$, -2 , $-114/7$ or -23 . For these special values one expects the existence of \mathcal{W}_3 invariant minimal CFT models. We will later be discussing the example of $c = 4/5$, for which a unitary \mathcal{W}_3 invariant CFT exists.

The decomposition of all currents in a \mathcal{W} -algebra in terms of quasi-primary currents and derivatives thereof can also be illustrated with the case of \mathcal{W}_3 . The first few (in terms of conformal dimension) quasi-primary currents are

$$T(z), \quad (TT)(z) - \frac{3}{10}\partial^2 T(z), \quad W(z), \quad (TW)(z) - \frac{3}{14}\partial^2 W(z), \quad \text{etc.} \quad (3.15)$$

3.2. Casimir algebras

In this section we present a generalization of the traditional Sugawara construction [317], see Section 2.3.2, which includes higher-spin generators in addition to the stress-energy tensor. This construction, which was first presented in [11], leads to so-called Casimir algebras, which are special examples of \mathcal{W} -algebras. In particular, the Casimir construction provides a realization of the \mathcal{W} -algebras associated with X_ℓ , where X_ℓ is one of the simply laced AKM algebras $A_\ell^{(1)}$, $D_\ell^{(1)}$ or $E_\ell^{(1)}$, with central charge given by $c = \text{rank}(X_\ell) = \ell$. More details will be provided in Chapter 7.

Our starting point is a conformal field $J(z)$, which takes values in a Lie algebra \mathfrak{g} (see Section 2.3.2 for our notations and conventions). We consider the so-called Casimir operators

$$W^{(s_i)}(z) = \frac{1}{s_i!} \eta^{(s_i)}(\mathfrak{g}, k) \sum_{a,b,c,\dots} d_{abc\dots} (J^a(J^b(J^c(\dots))))(z), \quad (3.16)$$

where $\eta^{(s_i)}(\mathfrak{g}, k)$ is some normalization constant and $d_{abc\dots}$ is a completely symmetric traceless \mathfrak{g} -invariant tensor of rank s_i . The index i labels a basis for the invariant tensors of \mathfrak{g} ; if \mathfrak{g} has rank ℓ we have $i = 1, 2, \dots, \ell$. The numbers s_i are equal to the so-called exponents of the Lie algebra \mathfrak{g} plus one. A list of exponents is provided in Appendix A.

Note that $T^{(s_i)} \equiv d^{abc\dots} T_a T_b T_c \dots$, with T_a as in (2.68), is the s_i -th Casimir operator of the Lie algebra \mathfrak{g} .

The first Casimir operator $T(z) \equiv W^{(2)}(z)$ is the Sugawara stress-energy tensor (2.72), which we discussed in Section 2.3.2. It satisfies the Virasoro OPE (2.3) with central charge $c(\widehat{\mathfrak{g}}, k)$ as given in (2.73). The fact that the d -symbols have been chosen to be traceless guarantees that the remaining Casimir operators $W^{(s_i)}(z)$, $i = 2, 3, \dots, \ell$, are primary fields of dimension s_i with respect to $T(z)$.

In the remaining part of this section we focus on the relatively simple example $\widehat{\mathfrak{g}} = A_2^{(1)}$, where two independent Casimir invariants of orders 2 and 3 exist. The third order Casimir operator takes the form

$$W^{(3)}(z) = \frac{1}{6} \eta^{(3)}(k) d_{abc} (J^a (J^b J^c))(z), \quad (3.17)$$

where

$$\eta^{(3)}(k) = \frac{1}{k+3} \sqrt{\frac{6}{5(2k+3)}}. \quad (3.18)$$

In addition to the aforementioned results that $T(z)$ satisfies the Virasoro operator algebra and that $W^{(3)}(z)$ is a primary field of dimension 3, we obtain the following contraction of $W^{(3)}$ with itself

$$\begin{aligned} \underbrace{W^{(3)}(z)W^{(3)}(w)} &= \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\ &+ \frac{1}{(z-w)^2} \left[2\beta\Lambda(w) + \frac{3}{10}\partial^2 T(w) + R^{(4)}(w) \right] \\ &+ \frac{1}{(z-w)} \left[\beta\partial\Lambda(w) + \frac{1}{15}\partial^3 T(w) + \frac{1}{2}\partial R^{(4)}(w) \right] \end{aligned} \quad (3.19)$$

where $\Lambda(z)$ and β are given in (3.6) and (3.7) and the central charge is equal to $c = c(A_2^{(1)}, k) = 8k/(k+3)$. The field $R^{(4)}(z)$ is a new primary field which cannot be expressed in terms of the second and the third order Casimir operators $T(z)$ and $W^{(3)}(z)$. For general k the field $R^{(4)}(z)$ does not vanish, which means that the operator algebra does not close on the space spanned by $T(z)$ and $W^{(3)}(z)$ (compare with our definition in Section 3.1).

This situation improves if the level k is chosen to be 1. It was shown in [11] that in that case the troublesome field $R^{(4)}(z)$ is a null field (corresponding to a null state in the Hilbert space), which decouples from the algebra. This then leads to the result that the $A_2^{(1)}$ Casimir algebra for $k = 1$ is actually *identical* to the \mathcal{W}_3 algebra (with $c = 2$) which we introduced in Section 3.1!

In Chapter 7 we shall show that the identification of the $c = 2$ \mathcal{W}_3 algebra with the level-1 $A_2^{(1)}$ Casimir algebra carries over to all of the simply-laced classical Lie algebras A_ℓ , D_ℓ and E_ℓ and the corresponding \mathcal{W} -algebras, which we will denote by $\mathcal{W}_c[X_\ell^{(1)}/X_\ell, 1]$ (see Appendix B). (Clearly, explicit calculations will not be possible in those cases and we will resort to different techniques.) We will then extend the Casimir construction to a coset construction, which will allow us to study the unitary minimal models of various \mathcal{W} -algebras.

3.3. \mathcal{W} -superalgebras; the example of super- \mathcal{W}_3

In this section we will take a first look at supersymmetric extensions of \mathcal{W} -algebras. Rather surprisingly, we will find that the ‘minimal’ supersymmetric extension of the \mathcal{W}_3 algebra is only consistent for two values, $c = \frac{10}{7}$ or $c = -\frac{5}{2}$, of the central charge, so that the minimal super- \mathcal{W}_3 algebra is what we call an ‘exotic’ algebra. The original derivation of this result was given in [201], where the associativity was checked by considering the crossing symmetry of 4-point functions. Here we will take the opportunity to illustrate (as in [4]) a different technique, which uses the Jacobi identity for graded commutators of field operators, see Section 2.2.

Before we turn to the ‘minimal’ super- \mathcal{W}_3 algebra, we make some remarks about the superconformal algebra. The $N = 1$ superconformal algebra is generated by the super stress-energy tensor $\widehat{T}(Z)$ of dimension $3/2$,

$$\widehat{T}(Z) = \frac{1}{2}G(z) + \theta T(z). \quad (3.20)$$

We follow the conventions of [145,142]: $Z = (z, \theta)$ is a complex supercoordinate, $T(z)$ is the ordinary stress energy tensor of dimension 2 and $G(z)$ is its fermionic superpartner of dimension $3/2$. The superconformal algebra is represented by the OPE

$$\widehat{T}(Z_1)\widehat{T}(Z_2) = \frac{1}{z_{12}^3} \frac{c}{6} + \frac{\theta_{12}}{z_{12}^2} \frac{3}{2} \widehat{T}(Z_2) + \frac{1}{z_{12}} \frac{1}{2} D\widehat{T}(Z_2) + \frac{\theta_{12}}{z_{12}} \partial\widehat{T}(Z_2) + \dots, \quad (3.21)$$

where $z_{12} = z_1 - z_2 - \theta_1\theta_2$, $\theta_{12} = \theta_1 - \theta_2$, $D = \partial_\theta + \theta\partial_z$, and $\partial = \partial_z$.

Let us now consider the superalgebra that is obtained by adding a primary supercurrent $\widehat{W}(Z)$ of dimension $5/2$ to the superconformal algebra. The fact that $\widehat{W}(Z)$ is primary of dimension $5/2$ is expressed by the OPE

$$\widehat{T}(Z_1)\widehat{W}(Z_2) = \frac{\theta_{12}}{z_{12}^2} \frac{5}{2} \widehat{W}(Z_2) + \frac{1}{z_{12}} \frac{1}{2} D\widehat{W}(Z_2) + \frac{\theta_{12}}{z_{12}} \partial\widehat{W}(Z_2) + \dots \quad (3.22)$$

Let us now consider the OPE $\widehat{W}(Z_1)\widehat{W}(Z_2)$. We suppose that we are in the ‘minimal’ case, where the operator product algebra for $\widehat{W}(Z)$ simply reads [201] (compare with (2.13))

$$[\widehat{W}] \cdot [\widehat{W}] = C[I] , \quad (3.23)$$

with $[I]$ denoting the superconformal family of the identity operator. In order to fix the coefficients in the OPE we consider the condition of associativity. We will use the Jacobi identities for ‘normal ordered graded commutators’ of the currents $\widehat{T}(Z)$ and $\widehat{W}(Z)$. These identities will allow us to fix various coefficients in a relatively easy way. The general graded Jacobi identity reads (compare with (2.47))

$$(-1)^{AC}[[A, B\}, C\}(Z) + \text{cycl.} = 0 \quad (3.24)$$

for general currents $A(Z)$, $B(Z)$, and $C(Z)$. With the the OPE’s (3.21) and (3.22) the Jacobi identities for $\widehat{T}\widehat{T}\widehat{T}$ and $\widehat{T}\widehat{T}\widehat{W}$ are guaranteed. For $\widehat{T}\widehat{W}\widehat{W}$ we have

$$[\widehat{T}, \{\widehat{W}, \widehat{W}\}](Z) + 2[\widehat{W}, \{\widehat{T}, \widehat{W}\}](Z) = 0. \quad (3.25)$$

The first few singular terms in $\widehat{W}(Z_1)\widehat{W}(Z_2)$ are easily fixed by writing $\widehat{W}(Z) = \frac{1}{\sqrt{6}}U(z) + \theta W(z)$ and using the OPE’s $W(z_1)W(z_2)$ and $U(z_1)U(z_2)$ as given in [342]. In combination with (3.25) this leads to

$$\begin{aligned} \widehat{W}(Z_1)\widehat{W}(Z_2) &= \frac{1}{z_{12}^5} \frac{c}{15} + \frac{\theta_{12}}{z_{12}^4} \widehat{T}(Z_2) + \frac{1}{z_{12}^3} \frac{1}{3} D\widehat{T}(Z_2) + \frac{\theta_{12}}{z_{12}^3} \frac{2}{3} \partial\widehat{T}(Z_2) + \frac{1}{z_{12}^2} \frac{2}{3} \widehat{T}^2(Z_2) \\ &+ \frac{\theta_{12}}{z_{12}^2} \frac{1}{(4c+21)} \left[(c + \frac{5}{2}) \partial^2 \widehat{T}(Z_2) + 22\widehat{T}D\widehat{T}(Z_2) \right] \\ &+ \frac{1}{z_{12}} \frac{1}{(4c+21)(10c-7)} \left[(36c+2)D\widehat{T}D\widehat{T}(Z_2) \right. \\ &\quad \left. + (2c^2 - c - 37)D\partial^2 \widehat{T}(Z_2) - (4c - 166)\widehat{T}\partial\widehat{T}(Z_2) \right] \\ &+ \frac{\theta_{12}}{z_{12}} \frac{1}{(4c+21)(10c-7)} \left[(112c - 160)\widehat{T}D\partial\widehat{T}(Z_2) \right. \\ &\quad \left. + \frac{4}{3} (2c^2 - 29c + 3)\partial^3 \widehat{T}(Z_2) + (144c + 8)D\widehat{T}\partial\widehat{T}(Z_2) \right] + \dots \end{aligned} \quad (3.26)$$

(Note that $\widehat{T}^2(Z)$ is actually the same as $\frac{1}{4}\partial D\widehat{T}(Z)$.)

The remaining associativity condition is the Jacobi identity for the triple product $\widehat{W}\widehat{W}\widehat{W}(Z)$. One finds

$$[\widehat{W}, \{\widehat{W}, \widehat{W}\}](Z) = \partial^2 \widehat{\Psi}(Z) , \quad (3.27)$$

where

$$\widehat{\Psi}(Z) = \frac{1}{9(4c+21)(10c-7)} \left[3(2c-83)\widehat{T}\partial D\widehat{W}(Z) + 12(18c+1)D\widehat{T}\partial\widehat{W}(Z) \right. \\ \left. - 6(2c-83)\partial\widehat{T}D\widehat{W}(Z) - 15(18c+1)\partial D\widehat{T}\widehat{W}(Z) \right. \\ \left. + (2c^2 - 29c + 3)\partial^3\widehat{W}(Z) \right]. \quad (3.28)$$

This result shows that the above operator algebra is *not* associative for generic values of the central charge c . However, one can check that the field $\widehat{\Psi}(Z)$ is superprimary, and hence null, for $c = 10/7$ or $c = -5/2$. We conclude that for these values of c all graded Jacobi identities are satisfied.

In [201], where the condition of associativity was analysed by considering crossing symmetry of 4-point correlators, it was shown that the ‘minimal’ super- \mathcal{W}_3 algebra, as given by the OPE’s (3.21), (3.22) and (3.26), is indeed associative for $c = 10/7$ and $c = -5/2$.

From the superspace OPE’s listed above the OPE’s of the component fields $T(z)$, $G(z)$, $W(z)$ and $U(z)$ can easily be obtained. One then finds that the bosonic \mathcal{W}_3 algebra (see Section 3.1) is a subalgebra of the ‘minimal’ super- \mathcal{W}_3 algebra for $c = 10/7$. A CFT model realizing super- \mathcal{W}_3 symmetry at $c = \frac{10}{7}$ will be discussed in Section 4.2.

We will come back to supersymmetric extensions of \mathcal{W} -algebras in general and of the \mathcal{W}_3 algebra in particular in later chapters.

4. \mathcal{W} -algebras and CFT

4.1. The chiral algebra in RCFT's.

We will not attempt to give a comprehensive review of the structure theory of rational conformal field theories (RCFT's), of which excellent accounts can be found elsewhere [259], but rather recall a few basic facts with special emphasis on the role played by the chiral algebra.

We first define what we mean by a *rational* CFT. (This notion was introduced by Friedan and Shenker in 1987 (unpublished).) A conformal field theory is called rational if it has the property that the matrix $\mathcal{N}_{h\bar{h}}$ which appears in the torus partition function (2.31) has finite rank. RCFT's have the property that their correlation functions on general (punctured) Riemann surfaces take the form of a finite sum of holomorphic times antiholomorphic expressions in the modular parameters of punctured surface. It was shown in [6] that in a RCFT the central charge c and the conformal dimensions h are all rational numbers.

An immediate consequence of the fact that $\mathcal{N}_{h\bar{h}}$ has finite rank is that the partition function Z (see (2.30)) of a RCFT can be written in the following form

$$Z = \sum_{(h,\bar{h})} (\chi_h + \dots)(\bar{\chi}_{\bar{h}} + \dots) , \quad (4.1)$$

where the dots stand for additional Virasoro characters (with increasing dimension) and the sum is over a *finite* set of labels (h, \bar{h}) . Because the unit operator occurs with multiplicity one in the theory, the label $(h, \bar{h}) = (0, 0)$ occurs precisely once in the above summation. The characters that are collected in the holomorphic factor $(\chi_0 + \chi_{s_2} + \dots)$ are all multiplied with $\bar{\chi}_0$. These terms in Z correspond to representations associated with chiral primary fields of dimension $(s_i, 0)$, which have the interpretation of currents corresponding to additional (chiral) symmetries in the CFT. The conformal families $[\phi_{(s_i, 0)}]$, together with the corresponding operator product expansions, define the so-called chiral algebra \mathcal{A} of the RCFT. (In those cases where the above prescription is ambiguous it will be assumed the characters in Z have been regrouped such that the chiral algebra is maximally extended.) The algebras \mathcal{A} and $\bar{\mathcal{A}}$, which is composed of the anti-chiral fields of dimension $(0, s_i)$, are extended conformal algebras. They contain a Virasoro subalgebra and additional currents which, due to modular invariance, all have integer conformal dimensions.

Let us now look at the other fields in the theory. It can be shown that these can be grouped into a finite set of representations of \mathcal{A} and $\overline{\mathcal{A}}$. These representations are HWM's. The highest weight state is annihilated by all the positive modes of both the Virasoro generators *and* the additional generators of the chiral algebra. The corresponding conformal field is called primary with respect to the chiral algebra. (See Chapters 6 and 7 for a more systematic description of the representation theory of extended conformal algebras.)

It was established in [324,257] that the fusion rules of the primary fields in a RCFT can be derived from the transformation properties under modular transformations of the characters $\hat{\chi}$ of the chiral algebra \mathcal{A} . This result indicates that the chiral algebra plays a central role: it dictates both the structure of the Hilbert space and the form of the interactions. These statements were made more precise in [94,258,259] where the following results for a RCFT with given chiral algebra \mathcal{A} were derived (the assumption is made that both chiral sectors have the same chiral algebra)

- i. all unitary representations of \mathcal{A} occur with multiplicity one,
- ii. the coupling of the left and the right chiral sectors is either diagonal or off-diagonal by an automorphism of the fusion algebra.

These results show that the classification of all RCFT's can in principle be done in a two-step process, where in the first step all chiral algebras that allow a finite set of unitary representations are determined and in the second step for each algebra the automorphisms of the fusion algebra are obtained.

It was shown in [79,80] that the chiral algebra of a RCFT contains an infinite set of Virasoro-primary currents if the central charge c satisfies $c \geq 1$. However, in many cases it has been found that the chiral algebra is finitely generated in the following sense. If we have a set of chiral currents we can always produce additional currents by taking derivatives and normal ordered products of currents in the set. We call a chiral algebra finitely generated if all its currents can be formed from a finite set of generating currents by repeatedly taking derivatives and normal ordered products. One easily checks that a chiral algebra which is finitely generated satisfies the defining properties of what we called a \mathcal{W} -algebra (Section 3.1). Thus we see the close connection between these two concepts.⁷

⁷ Caution: according to our definition, the chiral algebra of a level-1 WZW model for one of the ADE classical Lie algebras is (the enveloping algebra of) the full AKM algebra and not the Casimir algebra discussed in Section 3.2.

Among the main themes of this review paper are the close connections between certain \mathcal{W} -algebras on the one hand and classical or affine Lie algebras on the other. This then might suggest that the problem of classifying RCFTs can be completely solved by exploiting these relations. However, a classification of RCFTs along the lines discussed in this section would at least require a proper understanding of *all* finitely generated \mathcal{W} -algebras, which can be of generic (deformable) or exotic type. In contrast, the DS reduction scheme and the coset construction usually lead to \mathcal{W} -algebras that can be defined for generic central charge c . There is thus a missing link, which is a more systematic understanding of exotic \mathcal{W} -algebras. It is sometimes possible to construct exotic \mathcal{W} -algebras as extensions or truncations, which can only be defined for special c , of generic \mathcal{W} -algebras. We refer to Section 5.2.3 for some further comments about this.

The actual construction of the chiral algebras for RCFT's has been further analyzed in [298,299,203]. These papers introduced a technique based on the notion of *simple currents* and, related to that, of the *center* of a RCFT. Simple currents are special primary fields whose fusion rules with any other field contain just one term. This means in particular that they have well-defined monodromy properties with all fields, and this property can be used to define an abelian symmetry group which is called the center of the RCFT.

Let us assume that we are given a RCFT which is diagonal with respect to a certain chiral algebra. If the set of all primary fields with respect to this chiral algebra contains simple currents we can usually construct new, off-diagonal modular invariant partition functions. These new invariants either correspond to an automorphism of the fusion rules of the original chiral algebra or to an extension of the chiral algebra. In most cases, the actual construction of these new invariants resembles the orbifold constructions that are well-known in CFT and string theory.

The simple current modular invariants that correspond to fusion rules automorphisms have been completely classified in [152]. For theories with a center $(\mathbb{Z}_p)^k$, with p prime, a complete classification of *all* simple current invariants was presented in [153], see also [154].

4.2. Examples: \mathcal{W}_3 and super- \mathcal{W}_3 minimal models

In the previous section we discussed the general form of the torus partition function of RCFT's. We already mentioned in Section 2.2 that modular invariant partition functions

for the minimal unitary series (2.26) of central charges c have been classified in the so-called ADE classification [157,76,77,223]. In a similar way, all unitary superconformal models with central charge $c < 3/2$ have been classified in [75,221]. Their central charges are in the discrete series $c_m = \frac{3}{2}(1 - \frac{8}{m(m+2)})$, $m = 3, 4, \dots$

We will now take a closer look at two specific models which we take from the classifications just cited. By looking at their chiral algebra (or chiral superalgebra) we will argue that they actually possess extended symmetries, which will be \mathcal{W}_3 symmetry for the first model and super- \mathcal{W}_3 symmetry for the second. We present these explicit examples to illustrate the general structure that we discussed in Section 4.1.

For our first example we take a CFT of central charge $c = 4/5$, which corresponds to $m = 5$ in the unitary discrete series (2.26). For this central charge, two modular invariant partition functions exist. Here we focus on the so-called exceptional modular invariant, which corresponds to a CFT theory which is related to the 3-states Potts model at criticality. This partition function contains only a subset of the primary fields that are allowed by unitarity

$$Z = |\chi_0 + \chi_3|^2 + |\chi_{2/5} + \chi_{7/5}|^2 + 2 |\chi_{1/15}|^2 + 2 |\chi_{2/3}|^2, \quad (4.2)$$

where the subscripts denote the conformal dimensions. Of particular interest in this model is the occurrence of primary fields with dimensions (h, \bar{h}) given by $(3, 0)$ and $(0, 3)$, respectively. The other primary fields in the model are local with respect to these spin-3 fields. Explicitly we have the following operator product expansions [29]

$$\begin{aligned} \phi_{(3,0)} \cdot \phi_{(2/5,*)} &= [\phi_{(7/5,*)}] , & \phi_{(3,0)} \cdot \phi_{(7/5,*)} &= [\phi_{(2/5,*)}] , \\ \phi_{(3,0)} \cdot \phi_{(2/3,2/3)} &= [\phi_{(2/3,2/3)}] , & \phi_{(3,0)} \cdot \phi_{(1/15,1/15)} &= [\phi_{(1/15,1/15)}] , \end{aligned} \quad (4.3)$$

where the $*$ stands for $2/5$ or $7/5$ and $[\phi_{(2/3,2/3)}]$ and $[\phi_{(1/15,1/15)}]$ denote either one of the two conformal families with identical conformal dimensions.

These relations clearly show the existence of extended chiral symmetries in the model, which are generated by the spin-3 primary fields. The true symmetry algebra of this particular model is thus an extension of the conformal algebra, which is generated by the Virasoro generators and the spin-3 primary fields $\phi_{(3,0)}(z)$ and $\phi_{(0,3)}(\bar{z})$. One expects that the partition function (4.2) can be reexpressed in terms of characters $\hat{\chi}$ which are defined with respect to the extended chiral algebra (*i.e.* the subalgebra generated by $T(z)$ and

$\phi_{(3,0)}(z)$). The action (4.3) of the spin-3 fields on the other primary fields suggests that the partition function is the ‘diagonal’ sum of squared extended characters $\hat{\chi}$

$$Z = |\hat{\chi}_0|^2 + |\hat{\chi}_{2/5}|^2 + |\hat{\chi}_{1/15,+}|^2 + |\hat{\chi}_{1/15,-}|^2 + |\hat{\chi}_{2/3,+}|^2 + |\hat{\chi}_{2/3,-}|^2, \quad (4.4)$$

where + and – refer to the eigenvalues of the spin-3 operators.

Of course, the chiral algebra of this CFT is precisely the \mathcal{W}_3 algebra which we introduced in Section 3.1, and the structure sketched above has been confirmed by a detailed analysis of its representation theory and of the structure of its modular invariants. The value $c = 4/5$ is actually the lowest central charge that admits unitary \mathcal{W}_3 invariant CFT’s, and the modular invariant (4.4) is the first in a long list of \mathcal{W}_3 modular invariants that are known by now.

For our second example we consider a minimal superconformal field theory of central charge $c = 10/7$. We pick this value, which corresponds to $m = 12$ in the superconformal unitary series $c = c_m$ cited above, since we saw in Section 3.3 that precisely for this value the ‘minimal’ super- \mathcal{W}_3 algebra can consistently be defined. As we will see later, this value is also consistent with bosonic \mathcal{W}_3 symmetry, since it is the $m = 6$ position in the unitary \mathcal{W}_3 series $c_m = 2(1 - \frac{12}{m(m+1)})$.

It was proposed in [44] that the superconformal field theory with partition function

$$Z_{E_6, D_8}^{(N=1)} = \frac{1}{4} \sum_{s=1, \text{odd}}^{13} (|\chi_{1s}^{NS} + \chi_{5s}^{NS} + \chi_{7s}^{NS} + \chi_{11s}^{NS}|^2 + (\chi^{NS} \rightarrow \tilde{\chi}^{NS}) + |\chi_{4s}^R + \chi_{8s}^R|^2) \quad (4.5)$$

is a diagonal modular invariant of the minimal super- \mathcal{W}_3 algebra at $c = 10/7$. In here we write χ^{NS} and $\tilde{\chi}^{NS}$ for the characters (without and with the $(-1)^F$ insertion) in the Neveu-Schwarz sector and χ^R for the characters in the Ramond sector of the $N = 1$ superconformal algebra. The labels (rs) label the various highest weight states in both sectors.

In later papers [191,302], the representation theory of the super- \mathcal{W}_3 algebra and the branching rules from the super- \mathcal{W}_3 characters to ordinary superconformal characters have been worked out (see also [81]). In [302] some controversy surrounding this model was resolved and it was established that the partition function (4.5) can indeed be written as

follows in terms of characters of the super- \mathcal{W}_3 algebra

$$\begin{aligned}
2 Z_{E_6, D_8}^{(N=1)} &= \left(|ch_0^{NS}|^2 + |ch_{1/14}^{NS}|^2 + |ch_{5/14}^{NS}|^2 + |ch_{1/7,+}^{NS}|^2 + |ch_{1/7,-}^{NS}|^2 \right) \\
&\quad + (ch^{NS} \rightarrow \widetilde{ch}^{NS}) \\
&\quad + \frac{1}{4} \left(|ch_{1/14}^R|^2 + |ch_{5/14}^R|^2 + |ch_{3/2}^R|^2 + |ch_{9/14,+}^R|^2 + |ch_{9/14,-}^R|^2 \right).
\end{aligned} \tag{4.6}$$

This combination is precisely a ‘diagonal’ combination of all characters of the $c = \frac{10}{7}$ super- \mathcal{W}_3 algebra! (We wrote ch_h^{NS} , \widetilde{ch}_h^{NS} and ch_h^R for the super- \mathcal{W}_3 characters in the Neveu-Schwarz and Ramond sectors, respectively, and indicated the sign of the W_0 eigenvalue w by \pm .) For the actual derivation of this result some detailed knowledge about characters and modular invariants for the bosonic \mathcal{W}_3 algebra at $c = 10/7$ was used.

Thus we see that this second example, although technically more involved, is on the same footing as the simple $c = 4/5$ example discussed above. In both cases it turned out to be possible to learn about extensions of conformal symmetry by careful inspection of known modular invariant partition functions. These observations are independent from the detailed study of the operator algebra of the currents involved in the extended algebra, and they easily go beyond current algebras that can be constructed by hand. For example, it is straightforward to extend the examples discussed above to the bosonic \mathcal{W}_N algebra at $c = 2\frac{N-1}{N+2}$ and to the minimal super- \mathcal{W}_N algebra at $c_N = \frac{(3N+1)(N-1)}{2(2N+1)}$ [302,189,190]. Clearly, the latter algebras cannot easily be obtained in closed form.

5. Classification through direct construction

5.1. The method

In this chapter we discuss a variety of extended Virasoro algebras for which the algebra (in the form of (anti-)commutators or OPE's) is explicitly known. A small number of those have been found by hand and an additional number have been constructed with the help of computer power. In the Sections 3.1 and 3.3 we already showed two explicit examples, which were the \mathcal{W}_3 and super- \mathcal{W}_3 algebras, respectively.

In later chapters, where we will discuss systematic methods such as Drinfeld-Sokolov reduction and the coset construction, we will recover some of the algebras listed below. However, these systematic approaches at best give existence proofs for some of the algebras, and they certainly do not lead to explicit results for OPE's. Although these are not always needed, it is definitely useful to have available explicit and rigorous constructions of some of the simplest \mathcal{W} -algebras.

We mentioned before that the technical difficulty in constructing an extended Virasoro algebra with a given set of extra higher-spin fields, is to make sure that the resulting algebra is associative. In Section 2.2 we discussed three alternative characterizations of associativity, which are believed to be equivalent. In Section 3.1 we introduced the important distinction between 'generic' algebras (which are associative for all values of the central charge c), and 'exotic' algebras, which are associative for a finite number of c values only. Below we will discuss examples of both types.

Before we come to an overview of the results obtained, we would like to say more about the method of analysis. Let us first consider the analysis using crossing symmetry of 4-point functions. This method was first applied in [342], where it was used to fix the coefficients in the spin-5/2 and spin-3 extended conformal algebras. In [61], the behavior of four-point functions under crossing symmetry was analyzed by using counting arguments based on the conformal block decomposition of the 4-point functions of the currents in extended algebras of the type $\mathcal{W}(2, s)$. The analysis of crossing symmetry was further systematized in [68] (see also [188]).

A second possibility is to do the analysis by using Jacobi identities for modes of the quasi-primary fields that constitute the \mathcal{W} -algebra. To streamline these computations some general theory was developed in [68,58,225]. The starting point for this is the following form of the commutation relation of the modes ϕ_m^i and ϕ_n^j of two quasi-primary fields ϕ^i

and ϕ^j , which can be derived [68] from (2.14), (2.15)

$$[\phi_m^i, \phi_n^j] = \sum_k C^{ij}_k P(m, n; h_i, h_j, h_k) \phi_{m+n}^k + \gamma^{ij} \delta_{m+n} \binom{m+h_i-1}{h_i+h_j-1}, \quad (5.1)$$

where

$$P(m, n; h_i, h_j, h_k) = \sum_{r=0}^{h_i+h_j-h_k-1} \binom{m+h_i-1}{h_i+h_j-h_k-1-r} \frac{(-1)^r (h_i-h_j+h_k)_r (n+m+h_k)_r}{r! (2h_k)_r}, \quad (5.2)$$

with $(x)_r = \Gamma(x+r)/\Gamma(x)$. One then observes that the Jacobi identities for those quasi-primaries that can be written as composites of the generating fields are implied by those of the generating fields. This reduces the analysis to the generating fields of the \mathcal{W} -algebra, which have been called ‘simple fields’ in [58]. The extension of the formalism of [58] to the case of $N=1$ \mathcal{W} -superalgebras was presented in [55].

The most extensive and systematic work on the explicit construction of \mathcal{W} -algebras was presented in the papers [58] and [225]. These two papers, which largely overlap, give explicit results for a large number of \mathcal{W} -algebras with one or two higher-spin generators in addition to the spin-2 Virasoro generator.

5.2. Overview of results

In the list below we restrict ourselves to algebras for which all operator products (or, equivalently, commutation relations) are explicitly known and for which associativity has been established. Algebras whose existence is conjectured on the basis of extrapolation, general reasoning or wishful thinking are deferred to later chapters. For completeness, we also mention a few (linear and non-linear) superconformal extensions of the Virasoro algebra, which are not \mathcal{W} -algebras in the strict sense, but which fit naturally into the list.

We recall the notation convention introduced in Appendix B: by a \mathcal{W} -algebra of type $\mathcal{W}(2, s_2, s_3, \dots, s_n)$ we mean an algebra generated by the Virasoro generator $T(z)$ and additional primary currents of spins s_2, s_3, \dots, s_n . For a \mathcal{W} -extension of the N -extended superconformal algebra ($N=1, 2, 3$ or 4) with currents that are superfields of spins s_2, s_3, \dots , we will write $\mathcal{SW}^{(N)}(2 - \frac{N}{2}, s_2, \dots, s_n)$, where the first entry denotes the super stress tensor. Notice that these notations only specify a certain type of algebra; in particular, it is possible that distinct algebras with the same set of spins of the generating currents exist.

5.2.1. Generic, linear algebras

(i) The Virasoro algebra, given in (2.3) or (2.7).

(ii) The N -extended superconformal algebras ($N = 1, 2, 3, 4$).

The classical N -extended superconformal algebras, which contain affine $so(N)$ as a subalgebra, were first given in [1]. For $N = 4$, an additional algebra with only an affine $su(2)$ subalgebra, exists (the so-called ‘small’ $N = 4$ superconformal algebra). The algebras with $N \leq 3$ and the small $N = 4$ algebra possess a unique central extension. For the $so(4)$ extended $N = 4$ algebra two independent central extensions (corresponding to Schwarzian derivatives in $N = 4$ superspace) exist [300]. The $N = 4$ quantum algebras are thus parametrized by two central charges c and c' , or, suppressing one of the central terms, by c and the value of a deformation parameter α [300,312]. The linear superconformal algebras with $N \geq 5$ do not admit a central extension [300].

(iii) w_∞ , \mathcal{W}_∞ and $\mathcal{W}_{1+\infty}$.

We discuss some linear, infinitely generated \mathcal{W} -algebras. Some early references for these algebras are [17,45,282,283, 284,19,260,20,21]; useful reviews are for example [285,281,313].

The simplest infinitely generated \mathcal{W} -algebra, which has been named w_∞ [17], can be viewed as the algebra of area preserving diffeomorphisms of a two-dimensional cylinder. The algebra contains generators $w_m^{(s)}$ of spin $s = 2, 3, 4, 5, \dots$. The defining commutation relations are

$$[w_m^{(s)}, w_n^{(t)}] = [(t-1)m - (s-1)n]w_{m+n}^{(s+t-2)}. \quad (5.3)$$

The generators $w_m^{(2)}$ generate a (classical) Virasoro subalgebra. However, the standard central extension of the Virasoro algebra can *not* be extended to the full algebra w_∞ , which should therefore be viewed as a classical \mathcal{W} -algebra.

The algebra \mathcal{W}_∞ , which was first given in [282,283], is a deformation of w_∞ which is such that the standard central term in the Virasoro sub-algebra can be extended to the whole algebra. \mathcal{W}_∞ can thus be viewed as the quantum version of w_∞ . This has been nicely illustrated in the context of \mathcal{W} -gravity, where it was shown that the quantization of the classical theory of w_∞ gravity leads to a quantum theory based on \mathcal{W}_∞ [36].

Following [282,283], we denote the generators of \mathcal{W}_∞ of spin s by V_m^i , where $s = i + 2$, so that the index i ranges from 0 to ∞ . The defining commutation relations for \mathcal{W}_∞ can

then be written as

$$[V_m^i, V_n^j] = \sum_{l \geq 0} g_{2l}^{ij}(m, n) V_{m+n}^{i+j-2l} + c_i(m) \delta^{ij} \delta_{m+n} . \quad (5.4)$$

The structure constants $g_{2l}^{ij}(m, n)$ and the central terms $c_i(m)$ are completely fixed by the Jacobi identities and take the following form. For the central terms we have

$$c_i(m) = m(m^2 - 1)(m^2 - 4) \cdots (m^2 - (i + 1)^2) c_i , \quad (5.5)$$

where the central charges c_i are given by

$$c_i = \frac{2^{2i-3} i! (i + 2)!}{(2i + 1)!! (2i + 3)!!} c . \quad (5.6)$$

The structure constant $g_{2l}^{ij}(m, n)$ are expressed as

$$g_l^{ij}(m, n) = \frac{1}{2(l + 1)!} \phi_l^{ij} N_l^{ij}(m, n) , \quad (5.7)$$

where the N_l^{ij} are given by

$$N_l^{ij} = \sum_{k=0}^{l+1} (-1)^k \binom{l+1}{k} [i+1+m]_{(l+1-k)} [i+1-m]_k [j+1+n]_k [j+1-n]_{(l+1-k)} . \quad (5.8)$$

with $[x]_n = \Gamma(x + 1)/\Gamma(x + 1 - n)$. Finally, the ϕ_l^{ij} are given by

$$\phi_l^{ij} = \sum_{k \geq 0} \frac{(-\frac{1}{2})_k (\frac{3}{2})_k (-\frac{l}{2} - \frac{1}{2})_k (-\frac{l}{2})_k}{k! (-i - \frac{1}{2})_k (-j - \frac{1}{2})_k (i + j - l + \frac{5}{2})_k} , \quad (5.9)$$

where $(x)_n = \Gamma(x + n)/\Gamma(x)$. The w_∞ algebra can be recovered from \mathcal{W}_∞ by performing a contraction.

An alternative algebra, which contains a generator of spin 1 in addition to the spin 2, 3, ... generators of \mathcal{W}_∞ has been called $\mathcal{W}_{1+\infty}$ [284]. In addition, matrix generalizations of \mathcal{W}_∞ and $\mathcal{W}_{1+\infty}$ have been considered [21,273]. Supersymmetric extensions of \mathcal{W}_∞ have been worked out in [37,34,35]. The papers [34,35] introduce a deformation parameter λ , which is such that the algebra super- $\mathcal{W}_\infty(\lambda)$ can be truncated on various subalgebras for special choices of λ .

A $c = 2$ realization of \mathcal{W}_∞ , which can be viewed as a theory of \mathbb{Z}_∞ parafermions, was discussed in [20]. In [21] more general unitary representations, of central charge $c = 2p$,

$p = 1, 2, \dots$, were given. For a systematic discussion of the representation theory of various infinite \mathcal{W} -algebras, see for example [272].

In Section 5.3.4 we will discuss a non-linear extension of \mathcal{W}_∞ , which can be viewed as a universal \mathcal{W} -algebra.

5.2.2. Generic, non-linear algebras

(i) $\mathcal{W}(2, 3)$.

This is the \mathcal{W}_3 algebra, which we discussed in Section 3.1. Its OPE's are given in (2.3), (2.5) (with $h_W = 3$) and (3.5)-(3.7).

(ii) $\mathcal{W}(2, 4)$.

The explicit OPE's for this algebra, which was discussed in [61,182,345,58,225], are (2.3), (2.5) (with $h_W = 4$) and

$$\begin{aligned}
W(z)W(w) &= \frac{c/4}{(z-w)^8} + \frac{2T(w)}{(z-w)^6} + \frac{\partial T(w)}{(z-w)^5} \\
&+ \frac{1}{(z-w)^4} \left[\frac{3}{10} \partial^2 T(w) + 2\gamma \Lambda(w) \right] + \frac{1}{(z-w)^3} \left[\frac{1}{15} \partial^3 T(w) + \gamma \partial \Lambda(w) \right] \\
&+ \frac{1}{(z-w)^2} \left[\frac{1}{84} \partial^4 T(w) + \frac{5}{18} \gamma \partial^2 \Lambda(w) + \frac{24}{\mu} (72c + 13) \Omega(w) \right. \\
&\quad \left. - \frac{1}{6\mu} (95c^2 + 1254c - 10904) P(w) \right] \\
&+ \frac{1}{(z-w)} \left[\frac{1}{560} \partial^5 T(w) + \frac{1}{18} \gamma \partial^3 \Lambda(w) + \frac{12}{\mu} (72c + 13) \partial \Omega(w) \right. \\
&\quad \left. - \frac{1}{12\mu} (95c^2 + 1254c - 10904) \partial P(w) \right] \\
&+ C_{44}^4 \left\{ \frac{1}{(z-w)^4} W(w) + \frac{1}{(z-w)^3} \frac{1}{2} \partial W(w) \right. \\
&\quad + \frac{1}{(z-w)^2} \left[\frac{5}{36} \partial^2 W(w) + \frac{28}{3(c+24)} H(w) \right] \\
&\quad \left. + \frac{1}{(z-w)} \left[\frac{1}{36} \partial^3 W(w) + \frac{14}{3(c+24)} \partial H(w) \right] \right\} , \tag{5.10}
\end{aligned}$$

where

$$\begin{aligned}
\Lambda(w) &= (TT)(w) - \frac{3}{10}\partial^2 T(w) \\
\Omega(w) &= (\Lambda T)(w) - \frac{3}{5}(\partial^2 TT)(w) - \frac{1}{28}\partial^4 T(w) \\
P(w) &= \frac{1}{2}\partial^2 \Lambda(w) - \frac{9}{5}(\partial^2 TT)(w) + \frac{3}{70}\partial^4 T(w) \\
H(w) &= (TW)(w) - \frac{1}{6}\partial^2 W(w)
\end{aligned} \tag{5.11}$$

and

$$\gamma = \frac{21}{22 + 5c}, \quad \mu = (5c + 22)(2c - 1)(7c + 68). \tag{5.12}$$

The self-coupling constant C_{44}^4 , which is fixed by the requirement of associativity, is given by (this value was announced in [61] and confirmed in [58,225])

$$(C_{44}^4)^2 = \frac{1}{\mu} 54(c + 24)(c^2 - 172c + 196). \tag{5.13}$$

(We give this explicit form of the algebra mainly for the purpose of illustrating how rapidly the complexity of the OPE's increases with increasing spin of the generating currents. This has been the last algebra that we display in this explicit form.) This algebra is related to the Lie algebras B_2 and C_2 by Drinfeld-Sokolov reduction [61,226]. (Warning: the title of the second paper notwithstanding, the algebra $\mathcal{W}(2, 4)$ is *not* directly relevant for the level-1 B_2 WZW models and the level $(1, k)$ coset models based on this Lie algebra. For that we need an algebra of type $\mathcal{W}(2, 5/2, 4)$ (see below), which can be viewed as the Casimir algebra (in the sense of Section 3.2) of the superalgebra $B(0, 2)$ [330].)

(iii) $\mathcal{W}(2, 6)$.

The existence of a generic algebra with a spin-6 additional current was announced in [61], where it was also shown that algebras of type $\mathcal{W}(2, s)$ with s integer or half-integer and $s > 6$ do not exist for generic central charge c . The spin-6 algebra was explicitly constructed in [131]; these results were then confirmed in [58,225]. It is expected that this algebra is related to the Lie algebra G_2 by Drinfeld-Sokolov reduction [61,24].

(iv) $\mathcal{W}(2, 3, 4)$.

This algebra, which is explicitly given in [58,225], is the third, after Virasoro and \mathcal{W}_3 , of the \mathcal{W}_N algebras, which are of type $\mathcal{W}(2, 3, \dots, N)$. Their discovery in [109,110,111,11] was the first systematic extension of Zamolodchikov's construction of the \mathcal{W}_3 algebra. (These algebras have not been constructed explicitly for $N \geq 5$). Of all \mathcal{W} -algebras, these are the

ones that are most easily tractable and that have received the most attention. We will come back to this series in Chapters 6,7.

(v) $\mathcal{W}(2, 4, 6)$.

Solutions to the associativity conditions an algebra with spins 2,4,6 were found in [225]. One of these has been identified as the bosonic projection of the $N = 1$ superconformal algebra (see Section 5.3.2.).

(vi) $\mathcal{W}(2, 5/2, 4)$.

The existence of this algebra was announced in [111]; the explicit construction was given in [134], see also [3]. It is the second of in a series of algebras which are related [327,328] to a coset construction based on $B_N^{(1)}$ (see Chapter 7) and which have been studied [330] from the point of view of the quantum Drinfeld-Sokolov reduction of the superalgebras $B(0, N)$ (see Chapter 6). The higher algebras in this series are not known explicitly.

(vii) $\mathcal{SW}^{(1)}(3/2, 2)$, $\mathcal{SW}^{(1)}(3/2, 3/2, 2)$.

These are examples of generic \mathcal{W} -extensions of the superconformal algebra [235,133,55]. The self-coupling of the current \widehat{W}_2 in the algebra $\mathcal{SW}^{(1)}(3/2, 2)$ vanishes for $c = -6/5$, in agreement with the finding in [201] that the algebra without self-coupling is only associative for $c = -6/5$. More general $N = 1$ supersymmetric \mathcal{W} -algebras have been proposed [270,107, 235,198,130, 90,125], but these have not been worked out in closed form.

(viii) $N = 2$ super- \mathcal{W}_3 [271,295].

The $N = 2$ super- \mathcal{W}_n algebras were first considered at the classical level, where they arise from a Drinfeld-Sokolov reduction of the superalgebras $A(n - 1, n - 2)$ see [270,107, 235,198,129, 205,237,265]. The quantum algebra $\mathcal{SW}^{(2)}(1, 2)$ was first given in explicit form in [271], see also [295].

(ix) Nonlinear extended superconformal algebras.

The first examples of these are the $SO(N)$ and $U(N)$ Knizhnik-Bershadsky superconformal algebras [232,40], which are extended superconformal algebras with non-linear (quadratic) defining relations. As such they are very similar to the non-linear \mathcal{W} -algebras. They all admit non-trivial central extensions. In the papers [69,207] (see also [136]) further nonlinear superconformal algebras were constructed. Among them are two exceptional algebras based on the superalgebras $G(3)$ and $F(4)$, which have $N = 7$ and $N = 8$ supersymmetries, respectively.

5.2.3. Exotic algebras

Let us briefly mention some results for these. Algebras of type $\mathcal{W}(2, s)$, with $s > 2$ integer or halfinteger, have been studied by a number of authors [342,61,58,225,188]. Among these, only the algebras with $s = 3, 4, 6$ are generic; exotic algebras with $s = 5, 7, 8$ and $s = 5/2, 7/2, \dots, 15/2$ have been constructed. The representation theory of these algebras was analyzed in [323,104]. A similar program for \mathcal{W} -superalgebras of type $\mathcal{SW}^{(1)}(3/2, s)$, with s (half-)integer with $2 \leq s \leq 7/2$, has been carried out [132] (see also [187]). Among these, the only generic algebra is the one with $s = 2$ which we discussed above; of the exotic algebras the one with $s = 5/2$ (which we called the super- \mathcal{W}_3 algebra) was presented in detail in Section 3.3.

Some further examples: we mention algebras with two higher-spin primaries (for example, $\mathcal{W}(2, 4, 5)$ in [225]) or with a multiplet of higher spin primaries [224], for example $\mathcal{W}(2, 4, 4)$ at $c = 1$ or $c = \frac{656}{11}$ (see also [58,225]). \mathcal{W} -superalgebras of type $\mathcal{SW}^{(2)}(1, (\frac{3}{2})_+, (\frac{3}{2})_-)$ were considered in [202]. Further examples of \mathcal{W} -superalgebras can be found in [31,32,57].

Instead of listing still more examples, let us make some general remarks about the exotic algebras. Their existence can be understood as follows. If we consider a minimal model, of central charge c_0 , of some *generic* \mathcal{W} -algebra (which could for example be Virasoro), it may happen that the model contains chiral primary fields of integer conformal dimension. Such fields can then be added to the set of currents in the original \mathcal{W} -algebra, giving rise to an extension of it. The associativity of the resulting algebra is guaranteed, but only for central charge $c = c_0$. Thus we should expect that, in general, this construction gives rise to exotic algebras.

In Section 4.2, we already discussed two examples this enhancement of the symmetry algebra in specific minimal models, which were a $c = 4/5$ minimal model and a supersymmetric $c = 10/7$ minimal model. In the latter example, the enhanced algebra was the super- \mathcal{W}_3 algebra, which is exotic. Obviously, the principle can be used to generate many more examples of exotic algebras.

Taking an opposite point of view, one can try to view exotic \mathcal{W} -algebras as ‘truncations’ of generic \mathcal{W} -algebras. The idea here is that in certain minimal models of \mathcal{W} -algebras, one or more of the higher-spin currents may be ‘hidden’, giving rise to a chiral algebra which is smaller than the original \mathcal{W} -algebra and which only exists for some specific values

of c .

Let us give some examples to explain this idea. Borrowing some results from Chapter 7, we mention that there exists the E_6 Casimir algebra, which is of type $\mathcal{W}(2, 5, 6, 8, 9, 12)$. It is present in the level-1 E_6 WZW model, and in coset models based on $E_6 \oplus E_6/E_6$ at level $(1, k)$ for large enough k [12,328]. However, in the coset model with $k = 1$, which has central charge $c = 6/7$, only a truncation of this algebra, of type $\mathcal{W}(2, 5)$ appears. The latter algebra is exotic, and had been found in the systematic analysis in [61]. An even more dramatic example is the \mathcal{W} -algebra for the coset model for $E_8 \oplus E_8/E_8$ at level $(1, 1)$, with $c = 1/2$, which reduces to the Virasoro algebra. (Interestingly, in both these examples the full extended algebra can be recognized after perturbing the CFT's with a well-chosen relevant operator [344].) In [4] a generic extension of the exotic super- \mathcal{W}_3 algebra of Section 3.3 has been proposed.

Let us conclude this section by remarking that, although we have some handles for studying the exotic \mathcal{W} -algebras, they are clearly less tractable than the generic algebras, which can be studied systematically on the basis of Lie algebra theory.

5.3. Relating various algebras

5.3.1. Factoring out spin- $\frac{1}{2}$ fermions

In principle, one can consider \mathcal{W} -algebras with generators of spin $s \leq 2$. Adding spin-1 generators to the Virasoro algebra leads to an algebra which is a semi-direct product of the Virasoro algebra with an affine Kac-Moody algebra. Algebras with more than one spin-3/2 supercurrent typically include some spin-1 currents as well (as is the case in the Knizhnik-Bershadsky algebras cited above).

It is also possible to consider spin- $\frac{1}{2}$ fermions as generators for a (graded) \mathcal{W} -algebra. They occur naturally, for example, in the $N = 3, 4$ linear extended superconformal algebras of [1]. However, it was shown in [173] that the generators of an extended algebra including spin-1/2 fermions can always be redefined in such a way that the fermions decouple from the algebra. When applied to the $N = 3, 4$ extended superconformal algebras, this leads to the non-linear $SO(3)$ and $SO(4)$ extended superconformal algebras of Knizhnik-Bershadsky.

5.3.2. Twisted and projected \mathcal{W} -algebras

In the theory of affine Kac-Moody (AKM) algebras, the idea of twisting an algebra is well-known [214]. A twisted algebra can be defined if the underlying finite dimensional Lie algebra possesses a discrete symmetry (automorphism), which in the case of the AKM algebras is a \mathbb{Z}_2 or, in one case, a \mathbb{Z}_3 discrete symmetry. The \mathbb{Z}_2 twisted AKM algebras are $A_\ell^{(2)}$, $D_\ell^{(2)}$ and $E_6^{(2)}$, the unique \mathbb{Z}_3 twisted algebra is $D_4^{(3)}$.

The close relation between AKM algebras on the one hand and \mathcal{W} -algebras on the other, which will be discussed in the Chapters 6 and 7, suggests that we can consider twisted versions of \mathcal{W} -algebras as well. Indeed, we will see that both under Drinfeld-Sokolov reduction and under the coset construction the twisting of an AKM algebra can be ‘pulled back’ to a corresponding \mathcal{W} -algebra. A simple example is a \mathbb{Z}_2 twisting of the \mathcal{W}_3 algebra [184], which is based on the \mathbb{Z}_2 symmetry that sends $T(z)$ to $T(z)$ and $W(z)$ to $-W(z)$, and which assigns half-odd-integer modes to the current $W(z)$. More general twisted \mathcal{W} -algebras have been discussed in [185,186].

Once we have a twisted \mathcal{W} -algebra, we can consider the subset of all currents in the algebra which are integer-moded. In terms of the automorphism that caused the twisting, this is precisely the invariant subset. This projected \mathcal{W} -algebra contains the Virasoro algebra, and forms a new \mathcal{W} -algebra by itself. Let us for example consider the \mathbb{Z}_2 projected \mathcal{W}_3 algebra. It will be clear that its generators include at least $T(z)$ and $\Phi^{(6)}(z)$, where the spin-6 current $\Phi^{(6)}(z)$ is defined in (3.13). By looking at the characters we can identify additional currents and we find that the \mathbb{Z}_2 projected \mathcal{W}_3 algebra at least contains independent generators of spins 2, 6, 8, 10, 12. Similarly, we can identify independent generators of spins 2, 4, 6, 8, 10, 12 for the \mathbb{Z}_2 projected \mathcal{W}_4 algebra.

The torus partition functions for the twisted \mathcal{W} -algebras considered in [184,186] are such that the chiral currents that are odd under the automorphism drop out. As a consequence, the chiral algebras of these models are, in general, precisely the projected \mathcal{W} -algebras that we introduced here.

In a very similar way, we can consider the \mathbb{Z}_2 projection of any graded \mathcal{W} -algebra. For these there is a natural \mathbb{Z}_2 automorphism, which assigns odd \mathbb{Z}_2 parity to the currents of half-odd-integer spin and even parity to the integer-spin currents. The projected algebra contains all currents in the original algebra that have integer spin (in terms of states this would mean that we select all integer-spin states in the Neveu-Schwarz vacuum sector). As before we can try to find a set of generating currents for this set. This gives us a

bosonic \mathcal{W} -algebra, which we call the *bosonic projection* of the original graded algebra. This algebra acts as the chiral algebra of GSO projected models based on the original graded \mathcal{W} -algebra.

An interesting example is the $N = 1$ superconformal algebra, which is generated by a spin-3/2 supercurrent $G(z)$ in addition to the spin-2 stress-energy tensor $T(z)$, with contractions

$$\begin{aligned} \underbrace{T(z)G(w)} &= \frac{3/2G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} \\ \underbrace{G(z)G(w)} &= \frac{2c/3}{(z-w)^3} + \frac{2T(w)}{z-w} . \end{aligned} \tag{5.14}$$

It was shown in [62] that the bosonic currents

$$\begin{aligned} W^{(4)}(z) &= (G\partial G)(z) + \dots \\ W^{(6)}(z) &= (G\partial^3 G)(z) + \frac{94+5c}{14+c}(\partial G\partial^2 G)(z) + \dots \end{aligned} \tag{5.15}$$

(where the dots stand for expressions involving the fields $T(z)$ which are such that these fields become primary of spin 4 and 6) generate all integer-spin currents of the $N = 1$ superconformal algebra. The bosonic projection of the $N = 1$ superconformal algebra is thus of type $\mathcal{W}(2, 4, 6)$. For completeness we mention that for central charge $c = 7/10$ both currents $W^{(4)}(z)$ and $W^{(6)}(z)$ are null-fields, so that for that case the bosonic projection reduces to the Virasoro algebra without any extension. The $c = 1$ case has been discussed in [95].

5.3.3. Relations with parafermion algebras

Certain RCFT's can be conveniently described in terms of chiral algebras that contain currents of fractional spin. Such algebras are generally called parafermion algebras. It is interesting to compare this description with the purely bosonic formulation, and in particular to study the (bosonic) chiral algebras of these RCFT's. We will briefly discuss this for a number of examples.

We first mention the so-called \mathbb{Z}_N parafermions [113], which are defined in RCFT's of central charge $c_N = 2(N-1)/(N+2)$. It has been found [12], that the bosonic algebra underlying the model of \mathbb{Z}_N parafermions is the \mathcal{W}_N -algebra. More precisely, the \mathbb{Z}_N parafermion model turns out to be the smallest (in terms of central charge) unitary CFT with \mathcal{W}_N -symmetry (this will be made clear in Chapter 7, where we discuss the dis-

crete series of unitary CFT's with \mathcal{W}_N -symmetry). The relation between \mathbb{Z}_N parafermion algebras and the \mathcal{W}_N -algebra has been carefully studied in [263,264,85].

An alternative choice are the D_N parafermions, $N \geq 3$, which were introduced in appendix A of [113]. They each allow for a series of unitary CFT's, with central charge given by

$$c(k) = (N - 1) \left\{ 1 - \frac{N(N - 2)}{(k + N - 2)(k + N)} \right\}, \quad (5.16)$$

with k is positive, half-integer for $N = 3$ and positive, integer for $N > 3$. It was observed in [172] that these CFT's can all be obtained as coset CFT's for the cosets $\widehat{so}(N)_k \oplus \widehat{su}(N)_1 / \widehat{so}(N)_{k+2}$, where the subscripts denote the levels. (For $N = 3$ we identify $\widehat{so}(3)_k$ with $\widehat{su}(2)_{2k}$.) Thanks to the formulation of these theories as coset CFT's, their bosonic chiral algebra can be studied in a straightforward way. For the case $N = 3$ this will be worked out in Chapter 7, where we will find that the bosonic \mathcal{W} -algebra underlying the generic $N = 3$ model is of type $\mathcal{W}(2, 3, 4, 5, 6, 6)$.

Still other examples of CFT's where parafermionic symmetries can be identified are the diagonal cosets $\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}} / \widehat{\mathfrak{g}}$ of general level $(L, l; L + l)$. For the case of $\widehat{\mathfrak{g}} = \widehat{su}(2)$ it has been found [8,222,289] that the models with fixed L (but varying l) can be characterized by a parafermionic current algebra which, for $L \geq 2$, includes one or more currents of spin $1 + 2/(L + 2)$. For $L = 2$ the extra current is a fermionic supercurrent $G(z)$ of spin $3/2$, with OPE's as in (5.14). For $L = 4$ the current algebra contains two spin-4/3 currents [115], see also [289,290].⁸ The bosonic projections of the parafermionic current algebras for $L \geq 3$ can again be studied by using the formulation as coset CFT's.

5.3.4. $\widehat{\mathcal{W}}_\infty(k)$ as a universal structure

Given the multitude of finitely generated \mathcal{W} -algebras, there have been attempts to find a universal underlying structure, from which a number of finitely generated algebras could be obtained by reduction or truncation. One would expect that a universal \mathcal{W} -algebra contains all spins $s \geq 2$, and is thus an infinitely generated \mathcal{W} -algebra (see 5.2.1 (iii)).

In [240] it was shown that at $c = -2$ the \mathcal{W}_N algebras can be constructed by reducing the \mathcal{W}_∞ algebra. Similar reductions have been proposed in the context of $d = 2$ \mathcal{W} -gravity and for the \mathcal{W} -constraints in matrix models.

⁸ By using the fact that the embedding $\widehat{su}(2)_4 \subset \widehat{su}(3)_1$ is conformal, it can be seen that the coset CFT's for $\widehat{so}(3)_k \oplus \widehat{su}(3)_1 / \widehat{so}(3)_{k+2}$, which were discussed earlier, arise as special CFT's based on $\widehat{su}(2)_{2k} \oplus \widehat{su}(2)_4 / \widehat{su}(2)_{2(k+2)}$.

A very interesting proposal was made in the papers [338,22], which discuss a non-linear deformation of \mathcal{W}_∞ , called $\widehat{\mathcal{W}}_\infty(k)$. In [22] the quantum version of this algebra arises as the chiral algebra of the non-compact coset model $Sl(2, \mathbb{R})_k/U(1)$. By using the connection with (generalized) parafermions one finds that for $k = -N$ this algebra truncates to the \mathcal{W}_N algebra at central charge $c = 2(N - 1)/(N + 2)$. In the limit $k \rightarrow \infty$, the algebra reduces to \mathcal{W}_∞ .

It has been proposed in [22] that the algebra $\widehat{\mathcal{W}}_\infty(k)$ is the quantum version of the classical algebra \mathcal{W}_{KP} , which corresponds to the *second* hamiltonian structure in the hamiltonian formulation of the KP hierarchy [156,338,127]. In the same spirit, the linear $\mathcal{W}_{1+\infty}$ algebra has been identified as the Poisson bracket algebra for the *first* hamiltonian structure of the KP hierarchy [337,341]. The KP hierarchy can be reduced to the N^{th} generalized KdV hierarchy, and the idea is that in this reduction the algebras \mathcal{W}_{KP} and $\widehat{\mathcal{W}}_\infty(k)$ reduce to the classical and quantum \mathcal{W}_N algebras, respectively.

It thus appears that the algebra $\widehat{\mathcal{W}}_\infty(k)$ plays the role of ‘universal \mathcal{W} -algebra’ for the \mathcal{W}_N -algebras, whose structure is based on the Lie algebras A_{N-1} . It is expected that similar universal algebras can be obtained for the other series of Lie algebra based \mathcal{W} -algebras.

As was pointed out in [105,106,22], connections such as the ones described here may have important applications in string theory. The algebra $\widehat{\mathcal{W}}_\infty(k)$ at $k = 9/4$ is a symmetry of Witten’s black hole solution to two-dimensional string theory [335] and as such it provides an infinite number of conserved quantities. On the other hand, one may expect that the role played by the KdV hierarchy in the context of matrix models for $d \leq 1$ string theories, can be extended to a role played by the KP hierarchy, and the associated $\widehat{\mathcal{W}}_\infty(k)$ algebra, in a more general context.

6. Quantum Drinfeld-Sokolov reduction

6.1. Introduction

The most powerful method of constructing \mathcal{W} -algebras is through the so-called quantum Drinfeld-Sokolov (DS) reduction, in which one starts with an affine Lie algebra (or rather a suitable completion of its enveloping algebra), and reduces this algebra by imposing some constraint on the generators. At the classical level this procedure, which leads to the so-called Gelfand-Dikii algebras [155], was pioneered by Drinfeld and Sokolov [100], and then further explored by numerous groups [339,248,249,15,16,18,23,24,25].

Quantizing the classical description however poses serious problems. In a series of papers Fateev and Lukyanov [109,110,111] proposed to quantize the Gelfand-Dikii algebras by quantizing the free fields in the classical Miura transformation. However, this approach works well only for the Lie algebra A_n (and to some extent also for D_n). The observation that for A_n the \mathcal{W} -generators commute with a set of ‘screening charges’ led to an ad hoc construction of other \mathcal{W} -algebras as the centralizer of a suitable set of screening charges.

Rather than making a classical detour one can formulate the reduction as a quantum problem from the outset. Starting with an affine Lie algebra $\widehat{\mathfrak{g}}$ one imposes a set of constraints by means of the BRST procedure. The reduced algebra $\mathcal{W}[\widehat{\mathfrak{g}}, k]$ is defined as the cohomology of the BRST operator [30,42,92,126,118,137,138,228]. This is the approach we will take in the present chapter. We will argue that from this definition the description of $\mathcal{W}[\widehat{\mathfrak{g}}, k]$ (for the conventional DS-reduction) as the centralizer of a set of screening charges follows quite easily, and that, in preferred circumstances, a generating series for the generators of this centralizer may be found by proper quantization of the Miura transformation. Thus making contact with the work of Fateev and Lukyanov.

The BRST approach not only provides a proper definition of the quantum \mathcal{W} -algebra, but also a functor that maps modules of the affine Lie algebra to modules of the corresponding \mathcal{W} -algebra. Hence, the BRST approach is a convenient tool for the study of \mathcal{W} -algebra representations too. The functor was studied in detail in [139].

In the next section we will briefly recall some results that were obtained in the context of Toda field theory, which historically preceded the systematic study of \mathcal{W} -algebras from the point of view of DS reduction. The remainder of the chapter is then devoted to a detailed description of the quantum DS reduction.

6.2. Lagrange approach: constrained WZW and Toda field theories

The classical field equations of a Toda field theory are

$$\partial\bar{\partial}\phi = \sum_{i=1}^{\ell} \beta\alpha_i \exp(\beta\alpha_i \cdot \phi) . \quad (6.1)$$

The field $\phi(z, \bar{z})$ is a vector in the weight space of a (finite dimensional) semisimple Lie algebra \mathfrak{g} , of rank ℓ , for which $\{\alpha_i\}$ is a set of simple roots. These equations can be derived from the lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial\phi \cdot \bar{\partial}\phi - V_{\beta} , \quad V_{\beta} = \sum_{i=1}^{\ell} \exp(\beta\alpha_i \cdot \phi) . \quad (6.2)$$

Toda field theories are conformally invariant, both at the classical and at the quantum level. At the quantum level, the conformally improved energy momentum tensor satisfies the Virasoro current algebra (2.3), with the central charge given by [245]⁹

$$c(\beta) = \ell - 12 \left| \beta\rho - \frac{1}{\beta}\rho^{\vee} \right|^2 . \quad (6.3)$$

For $\mathfrak{g} = sl(2)$, the Toda field theory reduces to a Liouville theory for a single scalar field.

In a series of papers [49,50,51,52] the connection of Toda field theories to \mathcal{W} -algebras was first observed and worked out. In [49,50] it was shown that at the classical level a Toda field theory possesses a set of conserved currents, whose Poisson bracket algebra forms a classical \mathcal{W} -algebra. In [51,52] this result was extended to the quantum theory. Both at the classical and at the quantum level the \mathcal{W} -generators are multilinear expressions in the fields ϕ and their derivatives. Toda field theories thus lead to free field realizations of \mathcal{W} -algebras with adjustable central charge $c(\beta)$ as in (6.3).

For special values of the coupling constant β , the central charge can take the value of the one of the minimal RCFT models of the associated \mathcal{W} -algebra (see Section 6.4 and Chapter 7 for a systematic discussion). It was argued in [246] that even with this special choice of β the Toda field theory cannot be identified directly with a \mathcal{W} -minimal RCFT, the problem being that certain projections are needed and that the Toda spectrum leads only to a subset of the minimal model primary fields. The papers [246,247] introduced

⁹ The minus sign in front of the second term is due to an unusual choice for the vacuum or, alternatively, to the continuation $\beta \rightarrow i\beta$ in (6.2). See [246] for a discussion of this point.

so-called conformally extended Toda theories, whose quantum lagrangian (for \mathfrak{g} simply-laced) is obtained by replacing the potential V_β in (6.2) by the combination $V_\beta + V_{-1/\beta}$. It is argued in [246,247] that these conformally extended Toda theories provide a lagrangian realization of the \mathcal{W} -minimal CFT's.

The papers [23,24] discuss the fact that Liouville and Toda field theories can be obtained as conformally reduced WZW theories. This reduction can be viewed as a gauge procedure: the Toda theory is obtained as the gauge invariant content of a certain gauged WZW theory. We should stress that the gauging employed here is different from the gauging of a left-right diagonal subgroup, which leads to a lagrangian realization of a coset conformal field theory. Instead, the gauging that leads to a Toda field theory uses an upper triangular maximal nilpotent subgroup on the left hand side and a lower triangular one on the right hand side.

It has become clear that under the reduction that takes a WZW field theory into a Toda field theory the affine Lie algebra characterizing the WZW theory reduces to a \mathcal{W} -algebra (see, for example [25]). This reduction, which we call a (classical or quantum) Drinfeld-Sokolov reduction, can be studied in a more algebraic fashion without making explicit reference to the underlying Lagrange field theories. This is the approach we follow in Section 6.3.

The Toda field theory approach has been quite instrumental in the supersymmetric case, where various new \mathcal{W} -superalgebras have been found as symmetry algebras of supersymmetric Toda field theories [270,107,235,198,269]. In general, a super Toda field theory based on a basic Lie superalgebra for which all simple roots can be chosen to be fermionic is integrable and superconformally invariant. Its conserved currents generate a \mathcal{W} -superalgebra. Special choices are $osp(1|2)$, which leads to the (unextended) $N = 1$ super-Virasoro algebra and $sl(2|1)$, which gives the $N = 2$ super-Virasoro algebra. The central charges (as a function of the coupling constant) of a large class of quantum \mathcal{W} -superalgebras were listed in [107]. The connection between constrained supersymmetric WZW models and super Toda theories, which is a supersymmetric version of the DS reduction scheme, was worked out in [90,125].

Finally we note that Toda field theories play an important role in the discussion of \mathcal{W} -gravity (see Section 8.1), where they arise as effective quantum theories for the \mathcal{W} -gravity degrees of freedom in the conformal gauge.

6.3. Algebraic approach to DS reduction

6.3.1. \mathcal{W} -algebras from DS reduction

In general the quantum Drinfeld-Sokolov (QDS) reduction consists of the following steps. One starts with a triple $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi)$, consisting of an affine Lie algebra $\widehat{\mathfrak{g}}$, an affine subalgebra $\widehat{\mathfrak{g}}' \subset \widehat{\mathfrak{g}}$ and a 1-dimensional representation χ of $\widehat{\mathfrak{g}}'$.¹⁰ Next, one imposes the first class constraints $g \sim \chi(g), \forall g \in \widehat{\mathfrak{g}}'$, by means of the BRST procedure. The cohomology of the BRST operator Q on the set of normal ordered expressions in currents, ghosts and their derivatives (or, equivalently, the cohomology of Q on the Hilbert space associated to this set of fields) is what is called the Hecke algebra $H_Q(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi)$ of the triple $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi)$. In particular, the zeroth cohomology $H_Q^{(0)}(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi)$ is a subalgebra. This subalgebra is what we would like to call the \mathcal{W} -algebra $\mathcal{W}_{DS}[\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi]$ associated to the triple $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi)$. There is a small subtlety however. For special values of the level k of $\widehat{\mathfrak{g}}$, the cohomology $H_Q^{(0)}(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi)$ may be slightly bigger than for generic values of k ('generic' meaning here, and in the sequel, $k \notin -h^\vee + \mathcal{Q}_+$). We will therefore define $\mathcal{W}_{DS}[\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi]$ to be $H_Q^{(0)}(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi)$ for generic values of k . Since the operator product expansions of the \mathcal{W} -generators are algebraic in k we may then simply extend the definition of $\mathcal{W}[\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi]$ to all values of k by means of the acquired OPA (except, possibly, for a finite set of c -values where the OPE coefficients become singular). We also would like to remark that, for the conventional QDS reduction to be discussed below, one can show that the higher cohomologies $H_Q^{(i)}(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi)$, $i \neq 0$ vanish for generic k .

In order for $\mathcal{W}_{DS}[\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi]$ to be a \mathcal{W} -algebra in the sense described in Chapter 3, one has to suitably choose the triple $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi)$. We will describe how a generic class of such triples is obtained from $sl(2)$ embeddings [14,116,275,149] (for the corresponding hierarchies of integrable differential equations see [322,73,74,91,89,250]).¹¹

¹⁰ This assumption on χ may be relaxed. That involves the use of second class constraints which can, however, be transformed to first class constraints by introducing a set of auxiliary fields (see *e.g.* [43]). The first example of this type is the so-called $\mathcal{W}_3^{(2)}$ -algebra [280,41] of type $\mathcal{W}(1, \frac{3}{2}, \frac{3}{2}, 2)$ that can be considered as the bosonic analogue of the $N = 2$ superconformal algebra. We will not discuss this more general case here, but present some examples in Section 6.3.3. See also, for example, [294] for more general results in this direction.

¹¹ It has recently been argued in [72] that these correspond to the so-called reductive \mathcal{W} -algebras, *i.e.* \mathcal{W} -algebras for which the classical limit is positive definite. The Lie algebra \mathfrak{g} corresponds to a linear truncation, which can be made in the limit $c \rightarrow \infty$, of the vacuum

Let us first briefly discuss the classical case. Suppose we have an $sl(2)$ subalgebra $\{T^3, T^+, T^-\}$ of \mathfrak{g} . The adjoint representation of \mathfrak{g} decomposes into $sl(2)$ representations of spin $j_k, k = 1, \dots, p$, say.¹² Then we may write the current $J(z) = J_a(z)T^a$ as

$$J(z) = \sum_{k=1}^p \sum_{m=-j_k}^{j_k} U_{k,m}(z)T^{k,m}, \quad (6.4)$$

where $T^{k,m}$ corresponds to the generator of spin j_k and isospin m under the $sl(2)$ subalgebra. In particular we may put $T^{1,1} = T^+, T^{1,0} = T^3, T^{1,-1} = T^-$. The $sl(2)$ subalgebra $\{T^3, T^+, T^-\}$ can be characterized by a so-called “defining vector” δ (which we can choose to lie in the fundamental Weyl chamber), such that the $sl(2)$ root $\hat{\alpha}$ is given by $\hat{\alpha} = \delta/(\delta, \delta)$, and $T^3 = \delta \cdot H$. Take $\hat{\mathfrak{g}}'$ to be the affine Lie subalgebra $\hat{\mathfrak{n}}_+$ generated by all $U_{k,m}(z), m > 0$.

Now, one may impose the constraint

$$\chi_{DS}(U_{k,m}(z)) = \begin{cases} 1 & \text{for } (k, m) = (1, 1), \\ 0 & \text{for all other } (k, m) \text{ such that } m > 0. \end{cases} \quad (6.5)$$

Classically, this set of constraints generates enough gauge invariance to bring the constrained currents in the so-called lowest weight gauge¹³

$$J_{fix}(z) = T^+ + \sum_{k=1}^p U_{k,-k}(z)T^{k,-k}. \quad (6.6)$$

The Poisson bracket structure of the current algebra induces a Poisson bracket structure on the reduced space, and one can show that the algebra of the $U_{k,-k}(z), k = 1, \dots, p$ closes with respect to this induced Poisson structure. In particular, the algebra contains a Virasoro subalgebra generated by $T(z) = \frac{1}{2}\text{Tr}(J_{fix}(z)^2)$ with respect to which the fields $U_{k,-k}(z)$ are primary of conformal dimension $j_k + 1$.

In the quantum set-up very few results have been obtained so far for $sl(2)$ embeddings other than the principal embedding.¹⁴ This is clearly an issue that deserves further study.

preserving subalgebra (vpa) generated by the modes $\{\phi_m^i, |m| < h(\phi^i)\}$ for all quasi-primary fields $\phi^i(z)$. The $sl(2)$ subalgebra corresponds to the subalgebra of the vpa generated by $\{L_{-1}, L_0, L_1\}$.

¹² To get first class constraints we will restrict ourselves here to $sl(2)$ embeddings such that all $j_k \in \mathbb{Z}$.

¹³ Note that our conventions slightly differ from those of [14,322].

¹⁴ The most extensively studied algebra of this type is the Polyakov-Bershadsky algebra $\mathcal{W}_3^{(2)}$ that arises from the so-called $so(3)$ embedding in $sl(3)$ [280,41]. For this embedding $\underline{8} \rightarrow \underline{1} + \underline{2} + \underline{2} + \underline{3}$ which indeed leads to a \mathcal{W} -algebra of type $\mathcal{W}(1, \frac{3}{2}, \frac{3}{2}, 2)$ as mentioned before.

For this reason, and for the sake of simplicity, from now on we will be discussing only the ‘conventional’ QDS reduction which corresponds to the principal $sl(2)$ embedding in \mathfrak{g} .¹⁵

The defining vector of the principal $sl(2)$ embedding is the ‘dual Weyl vector’ ρ^\vee , *i.e.* for $\alpha \in \Delta_+$ we have $(\rho^\vee, \alpha) = 1$ if and only if α is a simple root of \mathfrak{g} . Denoting the currents corresponding to positive roots α by $e_\alpha(z)$, and choosing $T^{1,1} = \sum e^{\alpha_i}$, the constraint (6.5) is now explicitly given by

$$\chi_{DS}(e_\alpha(z)) = \begin{cases} 1 & \text{for simple roots } \alpha_i \\ 0 & \text{otherwise.} \end{cases} \quad (6.7)$$

We introduce pairs of ghost fields $(b_\alpha(z), c_\alpha(z))$, one for every positive root $\alpha \in \Delta_+$. The BRST-operator corresponding to the constraint (6.6) is given by $Q = \oint j_{BRST}(z) = Q_0 + Q_1$, where

$$Q_0 = \oint \frac{dz}{2\pi i} \left(\sum_{\alpha \in \Delta_+} (c_\alpha e_\alpha)(z) - \frac{1}{2} \sum_{\alpha\beta\gamma \in \Delta_+} f^{\alpha\beta\gamma} (b_\gamma c_\alpha c_\beta)(z) \right) \quad (6.8)$$

is the standard differential associated to $\widehat{\mathfrak{n}}_+$, $f^{\alpha\beta\gamma}$ are the structure constants of \mathfrak{n}_+ , and

$$Q_1 = - \oint \frac{dz}{2\pi i} \sum_{\alpha \in \Delta_+} (c_\alpha(z) \chi_{DS}(e_\alpha(z))) . \quad (6.9)$$

They satisfy

$$Q^2 = Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0 . \quad (6.10)$$

The algebra $\mathcal{W}[\widehat{\mathfrak{g}}, k]$ will, at least, contain the Virasoro algebra. An explicit representative is given by

$$T(z) = T^{Sug}(z) + \rho^\vee \cdot \partial h(z) + T^{gh}(z) , \quad (6.11)$$

where $T^{Sug}(z)$ is the Sugawara stress-energy tensor (2.72), and

$$T^{gh}(z) = \sum_{\alpha \in \Delta_+} (((\rho^\vee, \alpha) - 1)(b_\alpha \partial c_\alpha)(z) + (\rho^\vee, \alpha)(\partial b_\alpha c_\alpha)(z)) \quad (6.12)$$

is the ghost contribution. Note that the choice (6.11) assigns conformal dimensions $(1 - (\rho^\vee, \alpha), (\rho^\vee, \alpha))$ to the pair $(b_\alpha(z), c_\alpha(z))$. Of course, the ‘improvement terms’ in the expression (6.11) are obtained by the requirement that the BRST current $j_{BRST}(z) =$

¹⁵ The associated \mathcal{W} -algebra will be denoted by $\mathcal{W}_{DS}[\widehat{\mathfrak{g}}, \widehat{\mathfrak{n}}_+, \chi_{DS}]$, or simply by $\mathcal{W}[\widehat{\mathfrak{g}}, k]$ if no confusion can arise.

$j_0(z) + j_1(z)$ becomes a primary field of conformal dimension one, so that $[Q, T(z)] = 0$. (Implying, at the same time, that the constraints $e_\alpha(z) \sim \chi_{DS}(e_\alpha(z))$ become conformally invariant.)

The central charge of this Virasoro algebra is given by

$$\begin{aligned} c &= \frac{k \dim \mathfrak{g}}{k + \mathfrak{h}^\vee} - 12k|\rho^\vee|^2 - 2 \sum_{\alpha \in \Delta_+} (6(\rho^\vee, \alpha)^2 - 6(\rho^\vee, \alpha) + 1) \\ &= \ell - 12|\alpha_+\rho + \alpha_-\rho^\vee|^2, \end{aligned} \quad (6.13)$$

where we have introduced $\alpha_+\alpha_- = -1$, $\alpha_- = -\sqrt{k + \mathfrak{h}^\vee}$.

In the same spirit one would like to construct the other generators of $\mathcal{W}[\widehat{\mathfrak{g}}, k]$. This is clearly too complicated in general (see however [92] for \mathcal{W}_3). However, given the outcome (6.6) of the classical reduction, we can make the following general observation.

The Virasoro generator (6.11), which is in the BRST cohomology, corresponds to the component $U_{1,-1}(z)$ of the constrained and gauge fixed current $J_{fix}(z)$ in (6.6). Similarly one expects that the other components $U_{k,-k}(z)$ have a counterpart in the BRST cohomology, such that the constructed \mathcal{W} -algebra is generated by a set of currents of conformal dimension $e_i + 1$, where the set $\{e_i, i = 1, \dots, \ell\}$ (the set of ‘exponents’ of \mathfrak{g}) is the set of $sl(2)$ spins appearing in the decomposition of the adjoint representation of \mathfrak{g} . Note that the numbers $e_i + 1$ are precisely the orders of the independent Casimirs of \mathfrak{g} .

We will now argue that the BRST cohomology can be characterized as the centralizer of a set of screening charges (for more details and precise statements we refer to [118,119,137,138]). This will establish the connection with the approach of Fateev and Lukyanov [109,110,111]. The cohomology of $Q = Q_0 + Q_1$ can be calculated by means of the so-called spectral sequence technique (see *e.g.* [59]) applied to the double complex of Q_0 and Q_1 (see (6.10)). The spectral sequence is a systematic approach for calculating the cohomology of Q by starting from the cohomology of Q_0 and making successive higher order corrections. In this particular case this procedure stops (*i.e.* the ‘spectral sequence collapses’), for generic level k , after the second term, which implies that $H_Q(*) \cong H_{Q_1}(H_{Q_0}(*))$. It can be proved that the Q_0 -cohomology is generated by the fields $\tilde{h}_i(z)$ and $c_{\alpha_i}(z)$, where

$$\tilde{h}_i(z) = \alpha_+ \left(h_i(z) + \sum_{\alpha \in \Delta_+} (\alpha, \alpha_i^\vee) (b_\alpha c_\alpha)(z) \right). \quad (6.14)$$

One easily checks, for example, that the OPE of $j_0(z)$ with these fields is regular. In the next step, to calculate the cohomology of Q_1 on the space of fields generated by $\tilde{h}_i(z)$ and

$c_{\alpha_i}(z)$, it is convenient to bosonize $\tilde{h}_i(z) = \alpha_i^\vee \cdot i\partial\phi(z)$. Then, in Q_0 -cohomology, one can identify

$$c_{\alpha_i}(z) = \exp(-i\alpha_+ \alpha_i \cdot \phi(z)) = \tilde{s}_i^+(z) . \quad (6.15)$$

In particular, since there is no Q_0 -cohomology at negative ghost numbers, we have

$$H_{Q_1}^{(0)} = \text{Ker } Q_1 = \bigcap_{i=1}^{\ell} \text{Ker } \oint \tilde{s}_i^+(z) , \quad (6.16)$$

where the kernels are taken on the fields generated by $\tilde{h}_i(z)$. Therefore we reach the conclusion that, for generic level k , $\mathcal{W}[\widehat{\mathfrak{g}}, k]$ can be identified with the centralizer of the set of ‘screening charges’ $\oint \tilde{s}_i^+(z)$, ($i = 1, \dots, \ell$) on the space of polynomials (and derivatives thereof) in $i\partial\phi^i(z)$. We would like to stress that, although we set out as defining the algebra $\mathcal{W}[\widehat{\mathfrak{g}}, k]$ independent of any particular realization of $\widehat{\mathfrak{g}}$, that—as follows from the above discussion—the \mathcal{W} -algebra $\mathcal{W}[\widehat{\mathfrak{g}}, k]$ comes naturally equipped with a realization in terms of rank $\mathfrak{g} = \ell$ scalar fields $\phi^i(z)$. Let us, for future use, denote the Fock space of these scalar fields by \mathcal{F}_Λ , where Λ labels the eigenvalue of the scalar zero modes $\alpha_0^i = p^i$ on the Fock space vacuum $|\Lambda\rangle$.

It is not hard to construct the Virasoro generator from this description. This simply amounts to the construction of a conformal dimension two operator under which all screening operators (6.15) become primary of dimension one. One finds

$$T(z) = -\frac{1}{2}(\partial\phi \cdot \partial\phi)(z) - (\alpha_+\rho + \alpha_-\rho^\vee) \cdot i\partial^2\phi(z) . \quad (6.17)$$

It is a well-known fact that for $\mathfrak{g} = sl(2)$, at generic level k , this is a complete description of the centralizer (see *e.g.* [227] for a simple proof).

It follows that the eigenvalue h_Λ of L_0 (conformal dimension) of the Fock space vacuum $|\Lambda\rangle$ is given by (compare with (2.65))

$$h_\Lambda = \frac{1}{2}(\Lambda, \Lambda + 2(\alpha_+\rho + \alpha_-\rho^\vee)) . \quad (6.18)$$

At this point we would like to mention one crucial difference between the simply-laced case and the non simply-laced case. In the simply laced case, where $\rho^\vee = \rho$, there exists a special point, namely $\alpha_\pm = \pm 1$, for which the background charge term in (6.17) vanishes. This is exactly the point where we will make contact with the corresponding coset construction of the \mathcal{W} -algebra (see Chapter 7). For non simply-laced Lie algebras there does not exist

a point for which the background charge term vanishes, and there is no known coset construction that directly corresponds to $\mathcal{W}[\widehat{\mathfrak{g}}, k]$.

The form (6.17) of the stress-energy tensor, and the corresponding value (6.13) of the central charge are identical to the result obtained from a Toda field theory for the scalar fields $\phi^i(z, \bar{z})$ (see Section 6.2), where we identify $\beta = \alpha_+$. This illustrates the close connection, which we mentioned earlier, between the DS reduction scheme and Toda field theories.

Not all Lie algebras \mathfrak{g} give rise to distinct \mathcal{W} -algebras. We would now like to discuss a remarkable duality which implies, for instance, that the \mathcal{W} -algebras corresponding to B_n and C_n are essentially the same [119] (see also [227]). Given a finite dimensional (semi-simple) Lie algebra \mathfrak{g} , let \mathfrak{g}' be its ‘Langlands dual’ (obtained from \mathfrak{g} by inverting the arrows in the Dynkin diagram, *e.g.* $B'_n \cong C_n$). We can identify the Cartan subalgebras of \mathfrak{g} and \mathfrak{g}' by $\alpha'_i = \alpha_i^\vee / \sqrt{r^\vee}$, where r^\vee is the ‘dual tier number’ of \mathfrak{g} (the maximal number of edges connecting two vertices of the Dynkin diagram of \mathfrak{g}). This identification preserves the inner products between the roots. Now, the centralizer of the screening charge $\oint \exp(-i\alpha_+ \alpha_i \cdot \phi(z))$ is generated by the Virasoro element

$$T_i(z) = -\frac{1}{2(\alpha_i, \alpha_i)} (\alpha_i \cdot \partial\phi \alpha_i \cdot \partial\phi)(z) - \frac{1}{2} (\alpha_+ \alpha_i + \alpha_- \alpha_i^\vee) \cdot i\partial^2\phi(z) \quad (6.19)$$

and the Cartan subalgebra elements $\beta \cdot i\partial\phi(z)$ orthogonal to $\alpha_i \cdot i\partial\phi(z)$. However, one easily checks that this centralizer coincides with the centralizer of the charge

$$\oint \exp(-i\alpha_- \alpha_i^\vee \cdot \phi(z)) = \oint \exp(-i\alpha'_+ \alpha'_i \cdot \phi(z)), \quad (6.20)$$

where $\alpha'_+ = \alpha_- \sqrt{r^\vee}$. This proves that the \mathcal{W} -algebras related to \mathfrak{g} and its dual \mathfrak{g}' are isomorphic, provided one identifies $r^\vee(k + \mathfrak{h}^\vee(\mathfrak{g})) = (k' + \mathfrak{h}^\vee(\mathfrak{g}'))^{-1}$. Moreover, if one applies this theorem to a simply-laced Lie algebra \mathfrak{g} (for which $\mathfrak{g} \cong \mathfrak{g}'$, $r^\vee = 1$, $\alpha_i = \alpha_i^\vee$) one discovers that the \mathcal{W} -generators, which by definition commute with the charges corresponding to the screening operators $\tilde{s}_i^+(z)$, automatically commute with the charges corresponding to the ‘dual screening operators’ $\tilde{s}_i^-(z) = \exp(-i\alpha_- \alpha_i \cdot \phi(z))$.

Another extremely important property of $\mathcal{W}[\widehat{\mathfrak{g}}, k]$, which also follows easily by examining the various $sl(2)$ subalgebras as in (6.19), is the Weyl group invariance of the above construction.¹⁶ We claim that the eigenvalues $w_{s_i}(\Lambda)$ of the zero modes of the generators

¹⁶ The invariance under the finite Weyl group W can be considered a justification for the terminology \mathcal{W} -algebra [139].

$W^{(s_i)}(z)$ on the highest weight vector $|\Lambda\rangle$ of \mathcal{F}_Λ are invariant under the (shifted) action of the Weyl group W of \mathfrak{g} , *i.e.* under (see for example (6.18))

$$\Lambda \rightarrow w(\Lambda + \alpha_+\rho + \alpha_-\rho^\vee) - (\alpha_+\rho + \alpha_-\rho^\vee), \quad \forall w \in W. \quad (6.21)$$

To prove this, note that for every simple root α_i of \mathfrak{g} , the eigenvalue h_i of $T_i(z)$ in (6.19)

$$\frac{1}{4}(\Lambda + \alpha_+\alpha_i + \alpha_-\alpha_i^\vee, \alpha_i^\vee)(\Lambda, \alpha_i) \quad (6.22)$$

as well as the numbers (β, Λ) for $(\beta, \alpha_i) = 0$, are invariant under the shifted action of $w = r_i$, *i.e.* the reflection in the simple root α_i . We conclude that the $w_{s_i}(\Lambda)$ are invariant under the group generated by all these simple reflections, which is precisely the Weyl group of \mathfrak{g} .

To make the previous discussion somewhat more explicit we will now illustrate the above in the context of a free field realization of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$. Introduce a set of bosonic first order fields $(\beta^\alpha(z), \gamma^\alpha(z))$ of conformal dimension $(1, 0)$ for every positive root $\alpha \in \Delta_+$ of \mathfrak{g} , and a set of $\ell = \text{rank}(\mathfrak{g})$ scalar fields $\phi^i(z)$. Introduce the bosonic Fock spaces $\mathcal{F}_\Lambda^{\phi\beta\gamma} = \mathcal{F}_{\alpha_+\Lambda}^\phi \otimes \mathcal{F}^{\beta\gamma}$, where the vacuum $|\Lambda\rangle$ is labelled by the scalar zero modes as $p^i|\Lambda\rangle = \alpha_+\Lambda^i|\Lambda\rangle$. Then we have a realization of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ on $\mathcal{F}_\Lambda^{\phi\beta\gamma}$ with highest weight Λ and level k [325,117,64,65] (see Section 2.3.1 for the case of $sl(2)$). For our purposes it suffices to recall the explicit expression for the Cartan subalgebra generators in the Chevalley basis

$$h_i(z) = -\alpha_-(\alpha_i^\vee \cdot i\partial\phi(z)) + \sum_{\alpha \in \Delta_+} (\alpha, \alpha_i^\vee)(\gamma^\alpha\beta^\alpha)(z). \quad (6.23)$$

The Virasoro algebra acts on $\mathcal{F}_\Lambda^{\phi\beta\gamma}$ by means of the Sugawara construction, which in this specific realization takes the form

$$T^{Sug}(z) = -\frac{1}{2}(\partial\phi \cdot \partial\phi)(z) - \alpha_+\rho \cdot i\partial^2\phi(z) - \sum_{\alpha \in \Delta_+} (\beta^\alpha\partial\gamma^\alpha)(z). \quad (6.24)$$

This free field realization comes naturally equipped with a set of screening operators

$$s_i^\dagger(z) = (\beta^{\alpha_i}(z) + \dots) \exp(-i\alpha_+\alpha_i \cdot \phi(z)), \quad i = 1, \dots, \ell \quad (6.25)$$

where the dots stand for terms of higher order in $\beta\gamma$ -fields. These screening operators are primary fields of conformal dimension one (under (6.24)) and satisfy the important

property that their operator product expansion with the affine currents is at most a total derivative. Consequently, they can be used to construct intertwining operators between Fock space modules. Specifically, the operator

$$Q = \int_{\mathcal{C}} dz_1 \dots dz_n s_{i_1}^+(z_1) \dots s_{i_n}^+(z_n) \quad (6.26)$$

will be an intertwining operator, provided the contour is closed in the homology of the local system determined by the multivalued integrand of (6.26). We will come back to this point in Section 6.4.

In terms of the above free field realization the modified stress energy tensor (6.11) takes the following form

$$T(z) = T^\phi(z) + T^{\beta\gamma bc}(z) , \quad (6.27)$$

where

$$\begin{aligned} T^\phi(z) &= -\frac{1}{2}(\partial\phi \cdot \partial\phi)(z) - (\alpha_+\rho + \alpha_-\rho^\vee) \cdot i\partial^2\phi(z) \\ T^{\beta\gamma bc}(z) &= \sum_{\alpha \in \Delta_+} (((\rho^\vee, \alpha) - 1)(b^\alpha \partial c^\alpha + \beta^\alpha \partial \gamma^\alpha)(z) + (\rho^\vee, \alpha)(\partial b^\alpha c^\alpha + \partial \beta^\alpha \gamma^\alpha)(z)) . \end{aligned} \quad (6.28)$$

Similarly the field $\tilde{h}_i(z)$, in the cohomology of Q_0 , takes the form

$$\tilde{h}_i(z) = \alpha_i^\vee \cdot i\partial\phi(z) + \alpha_+ \sum_{\alpha \in \Delta_+} (\alpha, \alpha_i^\vee)(b^\alpha c^\alpha + \beta^\alpha \gamma^\alpha)(z) . \quad (6.29)$$

One can show, again by using a spectral sequence argument, that [42,92,126,118]

$$H_Q(\mathcal{F}_{\alpha_+\Lambda}^\phi \otimes \mathcal{F}^{\beta\gamma} \otimes \mathcal{F}^{bc}) \cong \mathcal{F}_{\alpha_+\Lambda}^\phi \otimes H_Q(\mathcal{F}^{\beta\gamma} \otimes \mathcal{F}^{bc}) \cong \mathcal{F}_{\alpha_+\Lambda}^\phi \quad (6.30)$$

where, in the first equality we have used the fact that Q does not depend on the scalar fields $\phi^i(z)$ and in the second that $H_Q(\mathcal{F}^{\beta\gamma} \otimes \mathcal{F}^{bc}) \cong \mathcal{C}$ which is, essentially, the familiar quartet decoupling mechanism. Using this isomorphism one can argue that, in fact, in Q -cohomology we have the following equivalences (see *e.g.* [42])

$$T(z) \sim T^\phi(z) , \quad \tilde{h}_i(z) \sim \alpha_i^\vee \cdot i\partial\phi(z) , \quad s_i^+(z) \sim \tilde{s}_i^+(z) . \quad (6.31)$$

This explains in more detail the origin of the equations (6.17) and (6.15).

Finally, it follows from (6.28), or equivalently from (6.11), that the conformal dimension of the Fock space representation $\mathcal{F}_{\alpha_+\Lambda}$ obtained by the reduction (6.30) is given by

$$h_{\alpha_+\Lambda} = \frac{1}{2}\alpha_+^2(\Lambda, \Lambda + 2\rho) - (\Lambda, \rho^\vee) = \frac{1}{2}(\alpha_+\Lambda, \alpha_+\Lambda + 2(\alpha_+\rho + \alpha_-\rho^\vee)) , \quad (6.32)$$

in agreement with (6.18).

The description of the \mathcal{W} -algebra as the centralizer of a set of screening charges can now also be proved without making use of the spectral sequence corresponding to the decomposition $Q = Q_0 + Q_1$ [138]. To this end one uses the isomorphism of fields and states in the characteristic Hilbert space (or ‘vacuum module’) \mathcal{H} of the \mathcal{W} -algebra (compare with Section 3.1). Then one constructs, for generic level k , a resolution $(C^{(i)}\mathcal{H}^g, d^{(i)})$ of the characteristic Hilbert space \mathcal{H}^g of $\widehat{\mathfrak{g}}$ in terms of free field Fock spaces. Recall that a resolution $(C^{(i)}L, d^{(i)})$ of a $\widehat{\mathfrak{g}}$ -module L is a complex $d^{(i)} : C^{(i)}L \rightarrow C^{(i+1)}L$, $d^{(i+1)}d^{(i)} = 0$ of $\widehat{\mathfrak{g}}$ -modules $C^{(i)}L$, where the differentials $d^{(i)}$ commute with the action of $\widehat{\mathfrak{g}}$, and is such that the cohomology of this complex is exactly the module L , ie

$$H_d^{(i)}(L) \cong \begin{cases} L & \text{if } i = 0 \\ 0 & \text{otherwise .} \end{cases} \quad (6.33)$$

The terms in the resolution of \mathcal{H}^g are labelled by elements of the Weyl group W of \mathfrak{g}

$$C^{(i)}\mathcal{H}^g \cong \bigoplus_{\{w \in W | \ell(w) = i\}} \mathcal{F}_{w\rho - \rho}^{\phi\beta\gamma} \quad (6.34)$$

and the differentials $d^{(i)}$ are operators of the type (6.26).

Applying the functor H_Q to the resolution (6.34), using (6.30), one obtains a resolution $(C^{(i)}\mathcal{H}, \tilde{d}^{(i)})$ of the characteristic Hilbert space \mathcal{H} of $\mathcal{W}[\widehat{\mathfrak{g}}, k]$ [138] (see also [267]), with terms

$$C^{(i)}\mathcal{H} \cong \bigoplus_{\{w \in W | \ell(w) = i\}} \mathcal{F}_{\alpha_+ + (w\rho - \rho)}^\phi \quad (6.35)$$

and differentials $\tilde{d}^{(i)}$ as in the complex (6.34), but with $s_i^+(z)$ replaced by $\tilde{s}_i^+(z)$ (see (6.31)). In particular the differential $\tilde{d}^{(0)}$ will be the collection of screening charges $\oint \tilde{s}_i^+(z)$ acting from $\mathcal{F}_0^\phi \rightarrow \mathcal{F}_{-\alpha_+ \alpha_i}^\phi$, and the intersection of its kernels is isomorphic to \mathcal{H} . So, again, we are led to the identification of $\mathcal{W}[\widehat{\mathfrak{g}}, k]$ with the centralizer of the set of screening charges $\oint \tilde{s}_i^+(z)$.

6.3.2. Character technique

In the above we have seen that, for generic values of the level k , the algebra $\mathcal{W}[\widehat{\mathfrak{g}}, k]$ can be identified with the centralizer of a set of screening charges $Q_i^+ = \oint \exp(-i\alpha_+ \alpha_i \cdot \phi(z))$. We have shown that $\mathcal{W}[\widehat{\mathfrak{g}}, k]$ contains at least the Virasoro algebra, and have given an explicit expression for the generating field (see (6.17)). We now want to discuss a useful

technique (which we will also often employ in Chapter 7) to get some insight into the other generators of $\mathcal{W}[\widehat{\mathfrak{g}}, k]$. This technique will be referred to as the ‘‘character technique’’ [63,62].

Suppose we have some \mathcal{W} -algebra of rank ℓ and type $\mathcal{W}(2, s_2, \dots, s_\ell)$. The corresponding Verma modules $M(\{h\}, c)$ are completely specified by the eigenvalues $h^{(s_i)}, i = 1, \dots, \ell$ of the zero modes $W_0^{(s_i)}$, and have a character (we will abbreviate $h^{(2)} = h$)

$$\text{ch}_M(q) = \text{Tr}_M q^{L_0 - \frac{c}{24}} = \frac{q^{h - \frac{c}{24}}}{\left(\prod_{k \geq 1} (1 - q^k)\right)^\ell}. \quad (6.36)$$

Conversely, given a character (6.36), we can anticipate the existence of a \mathcal{W} -algebra of rank ℓ . Equation (6.36), however, does not give us any information on the conformal dimensions s_i of the generators. In general, the Verma modules $M(\{h\}, c)$ will not be irreducible, due to the presence of singular vectors. The singular vector structure might be extremely complicated. However, there is one case where one knows some singular vectors beforehand. That is when we consider the Verma module built on the vacuum $|0\rangle$ of the conformal field theory (*i.e.* $h^{(s_i)} = 0, \forall i$). In that case, all the vectors

$$W_n^{(s_i)}|0\rangle, \quad n \geq -s_i + 1, \quad i = 1, \dots, \ell \quad (6.37)$$

will be singular (see (3.2)). These singular vectors generate a submodule $SM(0, c)$. Clearly, the factor module $\widetilde{M} = M(0, c)/SM(0, c)$ will have a character

$$\text{ch}_{\widetilde{M}}(q) = q^{-\frac{c}{24}} \prod_{i=1}^{\ell} \left(\frac{(1-q) \dots (1-q^{s_i-1})}{\prod_{k \geq 1} (1-q^k)} \right) = \frac{q^{-\frac{c}{24}}}{\prod_{i=1}^{\ell} (F_{s_i}(q))} \quad (6.38)$$

where we have introduced

$$F_s(q) = \prod_{k \geq s} (1 - q^k). \quad (6.39)$$

It may of course happen that the factor module $\widetilde{M} = M(0, c)/SM(0, c)$ still contains singular vectors, in which case the ‘true’ (*i.e.* irreducible) vacuum module \mathcal{H} (*i.e.* the characteristic Hilbert space) is even smaller. If, however, for generic c -values the generators $W^{(s_i)}(z)$ are independent, then (6.38), for generic c -values, will be the character of the vacuum module \mathcal{H} .¹⁷ Clearly, given the character (6.38), we can anticipate an underlying

¹⁷ It can happen that, even for generic c -values, the vacuum Verma module contains additional singular vectors beyond those displayed in (6.37). Suppose the lowest one occurs at L_0 -level N . Then $\text{ch}_{\mathcal{H}}$ will have the form (6.38) multiplied by a factor $(1 - a_N q^N - a_{N+1} q^{N+1} - \dots)$ where $a_k \in \mathbb{Z}_+$. This happens for instance in the bosonic projection of the $N = 1$ superconformal algebra [62] (see Section 5.3.2).

\mathcal{W} -algebra of type $\mathcal{W}(2, s_2, \dots, s_\ell)$.

Let us now apply these considerations to the QDS reduction. In the Section 6.3.1 we have mentioned the existence of a resolution of the characteristic Hilbert space \mathcal{H} in terms of Fock space modules (see (6.35)). This resolution allows us to compute the character of \mathcal{H} by means of the Euler-Poincaré principle. For $w \in W$ we have (see (6.32))

$$h_{\alpha_+(w\rho-\rho)} = -(w\rho - \rho, \rho^\vee) . \quad (6.40)$$

Thus, we obtain

$$\begin{aligned} \text{ch } \mathcal{H}(q) &= \sum_i (-1)^i \text{ch }_{C^{(i)}\mathcal{H}}(q) = q^{-\frac{c}{24}} \sum_{w \in W} \epsilon(w) \text{ch }_{\mathcal{F}_{\alpha_+(w\rho-\rho)}}(q) \\ &= \frac{q^{-\frac{c}{24}}}{\left(\prod_{k \geq 1} (1 - q^k)\right)^\ell} \sum_{w \in W} \epsilon(w) q^{-(w\rho-\rho, \rho^\vee)} \end{aligned} \quad (6.41)$$

which, by the Weyl-Kac denominator formula (see Appendix A)

$$\sum_{w \in W} \epsilon(w) e^{w\rho-\rho} = \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha}) , \quad (6.42)$$

can be rewritten as

$$\begin{aligned} \text{ch } \mathcal{H}(q) &= q^{-\frac{c}{24}} \frac{\prod_{\alpha \in \Delta_+} (1 - q^{(\rho^\vee, \alpha)})}{\left(\prod_{k \geq 1} (1 - q^k)\right)^\ell} = q^{-\frac{c}{24}} \frac{\prod_{i=1}^\ell (\prod_{k=1}^{e_i} (1 - q^k))}{\left(\prod_{k \geq 1} (1 - q^k)\right)^\ell} \\ &= \frac{q^{-\frac{c}{24}}}{F_{e_1+1}(q) \dots F_{e_\ell+1}(q)} . \end{aligned} \quad (6.43)$$

Here we have used the fact that ρ^\vee is precisely the defining vector of the principal $sl(2)$ subalgebra of \mathfrak{g} . Under this subalgebra the adjoint representation of \mathfrak{g} decomposes into $sl(2)$ representations of spin e_i , $i = 1, \dots, \ell$, where $\{e_i\}$ are the exponents of \mathfrak{g} . From the above discussion, we would thus expect $\mathcal{W}[\widehat{\mathfrak{g}}, k]$ to be a \mathcal{W} -algebra of type $\mathcal{W}(s_1, \dots, s_\ell)$ where the dimensions $s_i = e_i + 1$ of the generating fields are exactly the orders of the independent Casimirs of \mathfrak{g} (see appendix A for a list), in agreement with the classical result [100,24] and our intuitive reasoning in Section 6.3.1. Moreover, if we manage to construct a set of fields of conformal dimensions $s_i = e_i + 1$ in the intersection of the kernels of the screening charges and succeed in proving that, for generic values of k , they are independent, then the above analysis will prove the completeness of this set as well as

the closure of the algebra that they generate. This is the strategy we will employ in the examples of Section 6.3.3.

In the application of this ‘character technique’ to coset models of CFT, to be discussed in Chapter 7, we do not have a priori knowledge about the irreducible vacuum modules of the presupposed \mathcal{W} -algebras. Nevertheless, one can often still get useful information from the known characters of the irreducible highest weight modules of the affine Lie algebras $\widehat{\mathfrak{g}}$. As an illustration, consider again the QDS reduction. For \mathfrak{g} simply-laced, and at the special point $\alpha_{\pm} = \pm 1$, we can identify the screening operators $s_i^{\pm}(z)$ with the expressions for the simple root generators (in the Chevalley basis) of the vertex operator realization of $\widehat{\mathfrak{g}}$ at level $k = 1$.¹⁸ Thus, for this particular value of α_{\pm} , the centralizer of the screening charges $Q_i^{\pm} = \oint \tilde{s}_i^{\pm}(z)$ can be identified with the subset of normal ordered products of the affine currents and their derivatives that are singlets under the horizontal algebra \mathfrak{g} . This set of ‘singlets’ can be studied by decomposing the irreducible $\widehat{\mathfrak{g}}$ characters under \mathfrak{g} . This, for simply laced \mathfrak{g} , again leads to the expression (6.43) [63,62]. We will come back to this point in Chapter 7.

6.3.3. Examples

In this section we will present some general examples of \mathcal{W} -algebras obtained through the quantum Drinfeld-Sokolov reduction. Although the previous discussion has been restricted to bosonic Lie algebras, most of the results easily carry over to the case of Lie superalgebras. Instead of developing this here, we will just present two examples at the end of this section and refer to the original papers for more details.

$$— \mathcal{W}[A_n^{(1)}, k] —$$

Since the results for the Lie algebra $A_n \simeq sl(n+1)$ are the most complete we will treat this example in detail and use it as a reference for the other examples to be discussed later.

Let $\{\epsilon_i, i = 1, \dots, n+1\}$ be the set of weights of the vector representation of A_n , normalized such that $\epsilon_i \cdot \epsilon_j = \delta_{ij} - \frac{1}{n+1}$. They satisfy the constraint $\sum \epsilon_i = 0$. The simple roots of A_n are given by $\alpha_i = \epsilon_i - \epsilon_{i+1}$. Consider a set of currents $\{U_k(z)\}$, of conformal dimension k , defined through the generating expression (“quantum Miura

¹⁸ Of course, this identification holds up to cocycle factors. These, however, do not influence the structure of the centralizer.

transformation”).

$$\begin{aligned}
R_{n+1}(z) &= - \sum_{k=0}^{n+1} U_k(z) (\alpha_0 \partial)^{n+1-k} \\
&= ((\alpha_0 \partial_z - \epsilon_1 \cdot i \partial \phi(z)) \dots (\alpha_0 \partial_z - \epsilon_{n+1} \cdot i \partial \phi(z))) .
\end{aligned} \tag{6.44}$$

We have for example

$$\begin{aligned}
U_0(z) &= 1 \\
U_1(z) &= \sum_i \epsilon_i \cdot i \partial \phi(z) = 0 \\
U_2(z) &= - \sum_{i < j} ((\epsilon_i \cdot i \partial \phi)(\epsilon_j \cdot i \partial \phi))(z) + \alpha_0 \sum_i (i-1) \epsilon_i \cdot i \partial^2 \phi(z) \\
&= -\frac{1}{2} (\partial \phi \cdot \partial \phi)(z) - \alpha_0 \rho \cdot i \partial^2 \phi(z) = T^\phi(z) \\
U_3(z) &= \sum_{i < j < k} ((\epsilon_i \cdot i \partial \phi)(\epsilon_j \cdot i \partial \phi)(\epsilon_k \cdot i \partial \phi))(z) - \alpha_0 \sum_{i < j} (i-1) \partial((\epsilon_i \cdot i \partial \phi)(\epsilon_j \cdot i \partial \phi))(z) \\
&\quad - \alpha_0 \sum_{i < j} (j-i-1) ((\epsilon_i \cdot i \partial \phi)(\epsilon_j \cdot i \partial^2 \phi))(z) + \frac{1}{2} \alpha_0^2 \sum_i (i-1)(i-2) (\epsilon_i \cdot i \partial^3 \phi)(z)
\end{aligned} \tag{6.45}$$

where we have used $\sum_i \epsilon_i = 0$, $\sum_{i < j} \epsilon_i \otimes \epsilon_j = -\frac{1}{2} \mathbb{1}$, and $\sum_i i \epsilon_i = -\rho$.

Now consider the singular part in the OPE $R_{n+1}(z) \tilde{s}_i^\pm(w)$. We will show that it is a total derivative for each $i \in \{1, \dots, n\}$, which clearly implies that the fields $U_k(z)$ are in the centralizer of the screening charges Q_i^\pm . Since $\epsilon_j \cdot \alpha_i \neq 0$ only for $j \in \{i, i+1\}$, the only terms contributing to the singular part of the OPE are

$$\begin{aligned}
&(\alpha_0 \partial_z - \epsilon_i \cdot i \partial \phi(z)) (\alpha_0 \partial_z - \epsilon_{i+1} \cdot i \partial \phi(z)) e^{-i\alpha_\pm \alpha_i \cdot \phi(w)} \\
&= \left(\frac{-\alpha_\pm^2}{(z-w)^2} + (\alpha_0 \partial_z - \epsilon_i \cdot i \partial \phi(z)) \left(\frac{-\alpha_\pm}{z-w} \right) \right. \\
&\quad \left. + \left(\frac{\alpha_\pm}{z-w} \right) (\alpha_0 \partial_z - \epsilon_{i+1} \cdot i \partial \phi(z)) \right) e^{-i\alpha_\pm \alpha_i \cdot \phi(w)} \\
&= \left(\frac{-\alpha_\pm^2 + \alpha_0 \alpha_\pm}{(z-w)^2} + \frac{i\alpha_\pm (\epsilon_i - \epsilon_{i+1}) \cdot \partial \phi(w)}{z-w} \right) e^{-i\alpha_\pm \alpha_i \cdot \phi(w)} \\
&= -\partial_w \left(\frac{1}{z-w} e^{-i\alpha_\pm \alpha_i \cdot \phi(w)} \right)
\end{aligned} \tag{6.46}$$

provided $\alpha_\pm^2 - \alpha_0 \alpha_\pm = 1$, *i.e.* $\alpha_0 = \alpha_+ + \alpha_-$. This proves that

$$R_{n+1}(z) \tilde{s}_i^\pm(w) = -\partial_w \left(\frac{1}{z-w} R_{n+1}^i(z) e^{-i\alpha_\pm \alpha_i \cdot \phi(w)} \right) , \tag{6.47}$$

where $R_{n+1}^i(z)$ is defined as in (6.44) but with the ϵ_i and ϵ_{i+1} terms removed.

We would now like to argue that, for generic values of α_0 , the fields $U_k(z)$ are independent. This would, in view of the results of Section 6.3.2, establish that the $U_k(z)$ form a complete set of generators for $\mathcal{W}[A_n^{(1)}, k]$ as well as the closure of the algebra of $U_k(z)$'s. To this end let us compute the eigenvalues $u_k(\Lambda)$ of the zero modes $U_{k,0}$ on the highest weight state of the Fock space \mathcal{F}_Λ . If we can show that the resulting polynomials in $\theta_i = (\Lambda, \epsilon_i)$ are independent then this evidently will imply the independence of the generators $U_k(z)$. Applying the operator $R_{n+1}(z)$ (see (6.44)) to the highest weight state $|\Lambda\rangle$ we find

$$-\sum_{k=0}^{n+1} u_k(\Lambda) z^{-k} (\alpha_0 \partial)^{n+1-k} = (\alpha_0 \partial - \frac{1}{z}(\Lambda, \epsilon_1)) \dots ((\alpha_0 \partial - \frac{1}{z}(\Lambda, \epsilon_{n+1}))) . \quad (6.48)$$

By applying this differential operator to the monomials $z^j, j = 0, \dots, n+1$ we find the following recurrence relation for the eigenvalues $u_k(\Lambda)$

$$\sum_{k=0}^j \frac{j!}{(j-k)!} \alpha_0^k u_k(\Lambda) = (-1)^n \prod_{k=1}^{n+1} ((\Lambda + \alpha_0 \rho, \epsilon_k) + (\frac{1}{2}n - j)\alpha_0) , \quad (6.49)$$

where we have used $(\rho, \epsilon_k) = \frac{1}{2}(n+2-2k)$. Its solution is (see *e.g.* [47])

$$u_k(\Lambda) = (-1)^{k-1} \sum_{i_1 < \dots < i_k} \prod_{j=1}^k ((\Lambda, \epsilon_{i_j}) + (k-j)\alpha_0) . \quad (6.50)$$

Concretely, we have for example

$$\begin{aligned} u_2(\Lambda) &= - \sum_{i_1 < i_2} \theta_{i_1} \theta_{i_2} - \frac{1}{4} \binom{n+2}{3} \alpha_0^2 = \frac{1}{2}(\Lambda, \Lambda + 2\alpha_0 \rho) , \\ u_3(\Lambda) &= \sum_{i_1 < i_2 < i_3} \theta_{i_1} \theta_{i_2} \theta_{i_3} + (n-1)\alpha_0 \sum_{i_1 < i_2} \theta_{i_1} \theta_{i_2} + \binom{n+2}{4} \alpha_0^3 , \end{aligned} \quad (6.51)$$

where $\theta_i \equiv (\Lambda + \alpha_0 \rho, \epsilon_i)$. In general, as one can see from (6.49), the eigenvalues $u_k(\Lambda)$ are symmetric polynomials in the variables θ_i . Since the Weyl group of A_n acts as the permutation group S_{n+1} on the vectors ϵ_i , the numbers $u_k(\Lambda)$ are invariant under the (shifted) action of the Weyl group W of A_n , *i.e.* $u_k(\Lambda) = u_k(w(\Lambda + \alpha_0 \rho) - \alpha_0 \rho)$.

In particular, for $\alpha_0 = 0$ we discover that $u_k(\Lambda)$ is simply the standard (homogeneous) symmetric polynomial of order k in the variables θ_i . These are well-known to be independent. This fact can be used to prove the independence of the generators $U_k(z)$ in

an open neighborhood of $\alpha_0 = 0$, and hence the independence for generic α_0 (*i.e.* for a dense subset of \mathcal{C}) [262,328]. This proves, as remarked in Section 6.3.2, the closure of the algebra generated by the fields $U_k(z)$, $k = 2, \dots, n + 1$.

By explicit calculation it has been shown that the currents $U_k(z)$ satisfy an algebra with quadratic defining relations [241,111]

$$U_k(z)U_l(w) = \sum_{\kappa \geq 2} \frac{1}{(z-w)^\kappa} \sum_{p+q=k+l-\kappa} C_{kl}^{pq}(\kappa) (U_p(z)U_q(w)), \quad (6.52)$$

where the coefficients $C_{kl}^{pq}(\kappa)$ are algebraic in α_0 and therefore do not have any singularities.

Note, however, that the generators of the algebra in the Miura basis are not primary nor quasiprimary, in general. A basis for the \mathcal{W} -algebra consisting of (quasi)-primary fields $\tilde{U}_k(z)$ should be obtainable by deformation of the fields $U_k(z)$. The projection of $U_k(z)$ onto a quasi-primary field $\tilde{U}_k(z)$ is easily accomplished (using *e.g.* [58]), and only involves coefficients which are algebraic in α_0 . One has *e.g.*

$$\tilde{U}_3(z) = U_3(z) - \left(\frac{n-1}{2} \right) \alpha_0 \partial U_2(z), \quad (6.53)$$

with eigenvalue (use (6.51))

$$\tilde{u}_3(\Lambda) = \sum_{i_1 < i_2 < i_3} \theta_{i_1} \theta_{i_2} \theta_{i_3}. \quad (6.54)$$

In this case the resulting field $\tilde{U}_3(z)$ is, in fact, primary. In general, the possibility of further projection onto a set of primary field generators is still an important open problem, and has only been checked in low spin cases. The analogous problem in the classical case has been solved in [24]. Explicit formulas were obtained in [93]. Clearly, in the quantum case, the primary field projection may brake down for isolated values of c , as manifested by the occurrence of singularities in the operator product expansion coefficients in Chapter 4 [267,268]. This does not yet explain the singularity at $c = -22/5$ in the OPE of two dimension three fields (see (3.5), (3.7)), since the deformation (6.53) is purely algebraic. However, there is another potential source of singularities, namely, the ones that arise when the normalization factor required to bring the fields $\tilde{U}_k(z)$ to their standard normalization $\langle \tilde{U}_k | \tilde{U}_k \rangle = c/k$ becomes singular. Indeed, the normalization factor required for (6.53) becomes singular for $c = -22/5$. As a final remark we note that in terms of the primary fields $W^{(k)}(z)$ the defining relations of the algebra are no longer quadratic in general.

Explicit operator product expansions in the $\mathcal{W}[A_2^{(1)}, k]$ -case were given in (3.5), and those for $\mathcal{W}[A_3^{(1)}, k]$ can be found in [58,225].

Let us, for definiteness, display the generators of the $\mathcal{W}_3 \cong \mathcal{W}[A_2^{(1)}, k]$ algebra even more explicitly. Choose thereto an orthonormal basis with respect to which the simple roots have the following coordinates: $\alpha_1 = \sqrt{2}(1, 0)$, $\alpha_2 = \sqrt{2}(-\frac{1}{2}, \frac{1}{2}\sqrt{3})$. Then ¹⁹

$$\begin{aligned} T(z) &= U_2(z) = \frac{1}{2}(H^1 H^1 + H^2 H^2) - \frac{1}{2}\sqrt{2}\alpha_0 \left(\partial H^1 + \sqrt{3}\partial H^2 \right) \\ \tilde{U}_3(z) &= \frac{1}{3\sqrt{6}}(3H^1 H^1 H^2 - H^2 H^2 H^2) - \alpha_0 \left(\frac{1}{2}H^1 \partial H^1 + \frac{1}{3}\sqrt{3}H^2 \partial H^1 - \frac{1}{2}H^2 \partial H^2 \right) \\ &\quad + \frac{1}{4}\sqrt{2}\alpha_0^2(\partial^2 H^1 - \frac{1}{3}\sqrt{3}\partial^2 H^2), \end{aligned} \quad (6.55)$$

where we have introduced $H^i(z) = i\partial\phi^i(z)$. The normalized generator $W(z) = \sqrt{3\beta}\tilde{U}_3(z)$, where $\beta = 16/(5c + 22) = 2/(4 - 15\alpha_0^2)$, satisfies the algebra of (3.5). Note that normalization factor indeed becomes singular for $c = -22/5$.

$$- \mathcal{W}[D_n^{(1)}, k] -$$

For the Lie algebra $D_n \simeq so(2n)$ we proceed similarly. Let $\{\pm\epsilon_i, i = 1, \dots, n\}$ denote the weights of the vector representation of D_n , normalized such that $\epsilon_i \cdot \epsilon_j = \delta_{ij}$. The simple roots are given by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i = 1, \dots, n-1$ and $\alpha_n = \epsilon_{n-1} + \epsilon_n$. Define

$$R_n(z) = ((\alpha_0\partial_z - \epsilon_1 \cdot i\partial\phi(z)) \dots (\alpha_0\partial_z - \epsilon_n \cdot i\partial\phi(z))). \quad (6.56)$$

A similar calculation as for $\mathcal{W}[A_n^{(1)}, k]$ shows that the singular part of the OPE between $R_n(z)$ and $\tilde{s}_i^\pm(w)$ is a total derivative for $i \in \{1, \dots, n-1\}$. However,

$$\begin{aligned} &(\alpha_0\partial_z - \epsilon_{n-1} \cdot i\partial\phi(z))(\alpha_0\partial_z - \epsilon_n \cdot i\partial\phi(z))e^{-i\alpha_\pm \alpha_n \cdot \phi(w)} \\ &= \partial_w \left(\frac{1}{z-w} e^{-i\alpha_\pm \alpha_n \cdot \phi(w)} \right) + \frac{2\alpha_\pm}{z-w} e^{-i\alpha_\pm \alpha_n \cdot \phi(w)} (\alpha_0\partial_z). \end{aligned} \quad (6.57)$$

So, we conclude that for $i = n$ only the highest component $V_n(z) = R_n(z) \cdot 1$ is in the centralizer of all the screening charges. Upon taking the OPE of $V_n(z)$ with itself one clearly generates other currents in this centralizer. Explicitly [111,112],

$$V_n(z)V_n(w) = \frac{a_n}{(z-w)^{2n}} + \sum_{k=1}^{n-1} \frac{a_{n-k}}{(z-w)^{2(n-k)}} (U_{2k}(z) + U_{2k}(w)) \quad (6.58)$$

¹⁹ The expressions are related to those in [114] by a Weyl reflection.

for some fields $U_{2k}(z)$. It is convenient to choose $a_k = \prod_{j=1}^{k-1} (1 - 2j(j+1)\alpha_0^2)$. Then, for example,

$$U_2(z) = -\frac{1}{2}(\partial\phi \cdot \partial\phi)(z) - \alpha_0\rho \cdot i\partial^2\phi(z) = T^\phi(z) . \quad (6.59)$$

The eigenvalue $v_n(\Lambda)$ of the zero mode $V_{n,0}$ on the highest weight vector $|\Lambda\rangle \in \mathcal{F}_\Lambda$ follows from (6.56)

$$v_n(\Lambda) = (-1)^n \prod_{k=1}^n (\Lambda + \alpha_0\rho, \epsilon_k) \quad (6.60)$$

and is again invariant under the (shifted) action of the Weyl group $\Lambda \rightarrow w(\Lambda + \alpha_0\rho) - \alpha_0\rho$. By using (6.58) one may derive similar formulas for the eigenvalues $u_{2k}(\Lambda)$, and show explicitly their invariance under the shifted action of the Weyl group [111,112]. By exploiting these expressions one establishes the independence of the generators $\{V_n(z)\} \cup \{U_{2k}(z), k = 1, \dots, n-1\}$, exactly as in the case of $\mathcal{W}[A_n^{(1)}, k]$. Noting that the set of dimensions $\{2, 4, \dots, 2n-2, n\}$ indeed agrees with the orders of the independent Casimirs of D_n , this establishes the closure of the algebra generated by $V_n(z)$ and $U_{2k}(z)$.

$$- \mathcal{W}[B_n^{(1)}, k] -$$

In the case of the Lie algebra $B_n \simeq so(2n+1)$ one may again try to proceed similarly. Introduce vectors $\epsilon_i, i = 1, \dots, n$ as in the case of D_n , in terms of which the simple roots of B_n are given by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i = 1, \dots, n-1$, and $\alpha_n = \epsilon_n$. Introduce $R_n(z)$ as in (6.56). As before, the singular part of the OPE of $R_n(z)$ with $\tilde{s}_i^\pm(w)$ is a total derivative for $i = 1, \dots, n-1$, but

$$(\alpha_0\partial_z - \epsilon_n \cdot i\partial\phi(z))e^{-i\alpha_\pm\alpha_n\cdot\phi(w)} = \frac{\alpha_\pm}{z-w}e^{-i\alpha_\pm\alpha_n\cdot\phi(w)} . \quad (6.61)$$

Obviously, none of the components in $R_n(z)$ commutes with all the screening charges and it is not clear how to proceed. As argued in Section 6.3.2 one expects a \mathcal{W} -algebra of the type $\mathcal{W}(2, 4, 6, \dots, 2n)$, realized in terms of n scalar fields. In particular, for $n = 2$, one would expect to recover the (unique) algebra of type $\mathcal{W}(2, 4)$ (see Chapter 5). This has recently been confirmed by explicit construction of the two generators and verification of their operator product expansion [226]. For $n = 3$ it has been shown by explicitly constructing all fields in the centralizer of the screening charges of dimension less or equal to seven that the algebra is of type $\mathcal{W}(2, 4, 6, \dots)$ [227]. Their OPEs seem to lead to

the structure constants of the third $\mathcal{W}(2, 4, 6)$ algebra of [225], but no definite answer is available at the time of writing.²⁰

Inspection of (6.61) suggests another interesting possibility, to be discussed below.

$$— \mathcal{W}[B(0, n)^{(1)}, k] —$$

The free field realization of the Lie superalgebra $B(0, n)^{(1)}$ leads, similar to the bosonic case of Section 6.3.1, to a description of the \mathcal{W} -algebra $\mathcal{W}[B(0, n)^{(1)}, k]$ as the centralizer of a set of screening charges $s_i(z)$ on the Fock space of n scalar fields $\phi^i(z)$ and one fermionic field $\psi(z)$ [204]. The screening operators $s_i(z)$, $i = 1, \dots, n-1$ are as for $B_n^{(1)}$, while $s_n(z) = \psi(z)e^{-i\alpha_{\pm}\epsilon_n \cdot \phi(z)}$. The occurrence of the fermion in $s_n(z)$ has an important consequence. Considering the OPE

$$\begin{aligned} & (\alpha_0 \partial_z - \epsilon_n \cdot i \partial \phi(z)) \psi(z) \psi(w) e^{-i\alpha_{\pm}\epsilon_n \cdot \phi(w)} \\ &= \partial_w \left(\frac{-\alpha_{\mp}}{z-w} e^{-i\alpha_{\pm}\epsilon_n \cdot \phi(w)} \right) + \frac{1}{z-w} e^{-i\alpha_{\pm}\epsilon_n \cdot \phi(w)} (\alpha_0 \partial_z), \end{aligned} \quad (6.62)$$

we observe that the operator $U_{n+\frac{1}{2}}(z) = R_n(z) \cdot \psi(z)$ ($R_n(z)$ as in (6.56)), of conformal dimension $n + \frac{1}{2}$, is part of the centralizer. By taking the OPE of $U_{n+\frac{1}{2}}(z)$ with itself one generates a set of fields $\{U_{2k}(z), k = 1, \dots, n\}$ analogous to the $\mathcal{W}[D_n^{(1)}, k]$ -case. In particular,

$$U_2(z) = -\frac{1}{2}(\partial\phi \cdot \partial\phi)(z) - \alpha_0 \rho \cdot i \partial^2 \phi(z) - (\psi \partial \psi)(z). \quad (6.63)$$

It has been conjectured that $\{U_{n+\frac{1}{2}}(z)\} \cup \{U_{2k}(z), k = 1, \dots, n\}$ is a complete set of generators of $\mathcal{W}[B(0, n)^{(1)}, k]$ [111] (in this ref. the algebra was denoted by $\mathcal{W}B_n$). For additional details see also [204,327,329,43,134,3].

$$— \mathcal{W}[A(n, n-1)^{(1)}, k] —$$

Another interesting (super) \mathcal{W} -algebra is obtained from the Drinfeld-Sokolov reduction of the Lie superalgebra $A(n, n-1) \simeq sl(n+1, n)$. At the classical level this reduction was studied in [197,128,143,107,235,199,200] and was shown to give rise to an $N = 2$ \mathcal{W} -algebra. The simplest quantum case, $n = 1$, was analyzed in detail in [43] and was shown to lead to the $N = 2$ superconformal algebra. The general quantum case was analyzed in [205,206,265]. For $n = 2$ this algebra was explicitly constructed in [271,295]

²⁰ This particular $\mathcal{W}(2, 4, 6)$ algebra is however claimed to be inconsistent in [135,231].

by imposing the associativity constraints. This \mathcal{W} -algebra is supposedly the chiral algebra of the $\mathcal{CP}_n \simeq SU(n)/SU(n-1) \times U(1)$ Kazama-Suzuki model.

The affine super algebra $A(n, n-1)^{(1)}$ has a realization in terms of $2n$ free scalar fields $\phi^i(z)$, and conjugate pairs of spin $(0, 1)$ bosonic (fermionic) first order fields $(\gamma^\alpha(z), \beta^\alpha(z))$ $((\xi^\alpha(z), \eta^\alpha(z)))$ corresponding to the positive even (odd) roots $\alpha \in \Delta_+^0$ ($\alpha \in \Delta_+^1$) of $A(n, n-1)$. As in the case of $B(0, n)^{(1)}$ the constraint is second class. It can be treated as a first class constraint upon introducing $2n$ additional free (Majorana) fermionic fields $\psi^i(z)$. The $2n$ screening operators of $A(n, n-1)^{(1)}$ turn out to be BRST equivalent to the screening operators $s_i(z) = \alpha_i \cdot \psi(z) e^{i\alpha + \alpha_i \cdot \phi(z)}$, where $\alpha_1, \dots, \alpha_{2n}$ is a set of (fermionic) simple roots for $A(n, n-1)$. The corresponding \mathcal{W} -algebra can be identified with the centralizer of the corresponding screening charges on the (ϕ^i, ψ^i) -Fock space.

Analogous to the bosonic case $\mathcal{W}[A_n^{(1)}, k]$, the centralizer can be found by means of a (super) Miura transformation. The corresponding Lax operator is most easily described in $N = 1$ superfield language. Introduce a Grassmann variable θ , and a set of $2n$ superfields $\Phi^i(Z) = \phi^i(z) + i\theta\psi^i(z)$, $Z = (z, \theta)$ as well as a superderivative $D_Z = \partial_\theta + \theta\partial_z$. Let

$$\begin{aligned} R_n(Z) &= \sum_{k=0}^{2n+1} U_{\frac{k}{2}}(Z) (\alpha_- D)^{2n+1-k} \\ &= ((\alpha_- D + \epsilon_1 \cdot iD\Phi(Z)) \dots (\alpha_- D + \epsilon_{2n+1} \cdot iD\Phi(Z))) . \end{aligned} \quad (6.64)$$

The ϵ_i are given by $\epsilon_i = (-1)^i(\lambda_i - \lambda_{i-1})$, where the λ_i , $i = 1, 2, \dots, 2n$, are the fundamental weights satisfying $\lambda_i \cdot \epsilon_j = \delta_{ij}$ and $\lambda_0 = \lambda_{2n+1} = 0$. In the superfield language the screening operators take the form $e^{i\alpha + \alpha_j \cdot \Phi}$. The singular terms in the OPE of the Lax operator $R_n(Z)$ with one of the screening operators arise from

$$\begin{aligned} &(\alpha_- D_1 + \epsilon_j \cdot iD\Phi(Z_1)) (\alpha_- D_1 + \epsilon_{j+1} \cdot iD\Phi(Z_1)) e^{i\alpha + \alpha_j \cdot \Phi(Z_2)} \\ &= (-1)^{j+1} D_2 \left(\frac{\theta_{12}}{Z_{12}} e^{i\alpha + \alpha_j \cdot \Phi(Z_2)} \right) , \end{aligned} \quad (6.65)$$

where we wrote $\theta_{12} = \theta_1 - \theta_2$, $Z_{12} = z_1 - z_2 - \theta_1\theta_2$ and we used $\alpha_+ \alpha_- = -1$. We see that all $U_{k/2}(Z)$, $k = 0, \dots, 2n+1$ are in the centralizer of the screening charges. The generators $U_0(Z)$ and $U_{1/2}(Z)$ are constant and it can be verified that $U_1(Z)$ and $U_{3/2}(Z)$ generate an $N = 2$ superconformal algebra of central charge

$$c = 3n \left(1 - (n+1)\alpha_-^2 \right) . \quad (6.66)$$

We thus obtain a \mathcal{W} -algebra of type $\mathcal{SW}^{(1)}(1, \frac{3}{2}, \dots, \frac{2n+1}{2})$ or, in terms of its $N = 2$ content, an algebra of type $\mathcal{SW}^{(2)}(1, 2, \dots, n)$.

$$- \mathcal{W}[\mathfrak{g}^{(r)}, k] -$$

The twisted affine Kac-Moody algebra $\mathfrak{g}^{(r)}$, $r = 2, 3$ can be obtained as the invariant sector under an outer automorphism τ of \mathfrak{g} of order r [214]. The corresponding QDS reductions $\mathcal{W}[\mathfrak{g}^{(r)}, k]$ can be determined from the commutative diagram

$$\begin{array}{ccc} \mathfrak{g}^{(1)} & \xrightarrow{\tau} & \mathfrak{g}^{(r)} \\ \downarrow d & & \downarrow d \\ \mathcal{W}[\mathfrak{g}^{(1)}, k] & \xrightarrow{\tau} & \mathcal{W}[\mathfrak{g}^{(r)}, k] \end{array} \quad (6.67)$$

i.e. they are obtained by twisting the algebra $\mathcal{W}[\mathfrak{g}^{(1)}, k]$.

As an example consider $\mathfrak{g} = A_2$. The outer automorphism τ of order two acts on the Cartan subalgebra generators in the Cartan-Weyl basis as

$$\tau(H^1) = -\frac{1}{2}H^1 + \frac{1}{2}\sqrt{3}H^2, \quad \tau(H^2) = \frac{1}{2}\sqrt{3}H^1 + \frac{1}{2}H^2. \quad (6.68)$$

By using (6.55) one can explicitly check

$$\tau(U_2(z)) = U_2(z), \quad \tau(\tilde{U}_3(z)) = -\tilde{U}_3(z). \quad (6.69)$$

The subset of invariant fields under the automorphism τ is thus exactly the projected \mathcal{W}_3 -algebra mentioned in Section 5.3.2. This algebra, and its generalizations, were studied in detail in [184,185,186].

6.4. Representation theory

6.4.1. The Kac determinant

An important tool in the investigation of the structure of Verma modules and their irreducible quotients is the so-called Kac determinant [212].

The hermiticity requirement $W_n^{(s)\dagger} = W_{-n}^{(s)}$, together with the normalization $\langle \Lambda | \Lambda \rangle$, uniquely define a symmetric sesquilinear form (“Shapovalov form”) on the Verma module $M(\Lambda, c)$. If $\{|v_i\rangle, i = 1, \dots, p_\ell(N)\}$ is a basis of $M(\Lambda, c)_{(N)}$ then the determinant of the $p_\ell(N) \times p_\ell(N)$ matrix $(\mathcal{M}_{(N)})_{ij} = \langle v_i | v_j \rangle$ is called the Kac determinant (compare with the discussion in Section 2.1). Here, $p_\ell(N)$ is the number of partitions of N on ℓ colors, *i.e.* $\sum p_\ell(N)t^N = (\prod(1 - t^k))^{-\ell}$.

Since the Kac determinant is clearly a polynomial in Λ , it is determined (up to a Λ independent factor) by its zero’s, which correspond to singular vectors in $M(\Lambda, c)$. The

strategy now is to construct explicitly a sufficient number of singular vectors. Instead of constructing singular vectors in the Verma modules directly, it is much easier to construct them in the Fock space representations. Since to every singular vector in \mathcal{F}_Λ corresponds a singular vector in $M(\Lambda)$ of the same weight, this also tells us something about the singular vectors in $M(\Lambda)$.

Singular vectors in \mathcal{F}_Λ are constructed by the following trick [321]. Suppose we have an intertwiner $Q : \mathcal{F}_\Lambda \rightarrow \mathcal{F}_{\Lambda'}$ (*i.e.* an operator commuting with the action of the \mathcal{W} -generators) then the image of $|\Lambda\rangle$ under Q is either zero or a singular vector in $\mathcal{F}_{\Lambda'}$. Intertwiners Q can be constructed out of properly integrated products of screening operators $\tilde{s}_i^+(z)$ as

$$Q = \int_{\mathcal{C}} dz_1 \dots dz_n \tilde{s}_{i_1}^+(z_1) \dots \tilde{s}_{i_n}^+(z_n) . \quad (6.70)$$

Since the OPE's of the \mathcal{W} -generators with $\tilde{s}_i^+(z)$ are at most a total derivative (by the definition of the \mathcal{W} -algebra), the expression (6.70) gives an intertwiner provided the contour \mathcal{C} is closed in the homology of the local system determined by the integrand of (6.70) (see *e.g.* [67] for more details). For the purpose of determining the Kac determinant it suffices to consider intertwiners built out of a single screening operator $\tilde{s}_i^+(z)$ only. Doing this, one finds that for weights Λ of the form $\Lambda = \alpha_+ \Lambda^{(+)} + \alpha_- \Lambda^{(-)}$, where $\Lambda^{(+)} \in P_+$, $\Lambda^{(-)} \in P_+^\vee$, the Fock space \mathcal{F}_Λ has a singular vector at level $N = (\Lambda^{(+)} + \rho, \alpha_i^\vee)(\Lambda^{(-)} + \rho^\vee, \alpha_i)$ for each $i = 1, \dots, \ell$. This singular vector and its \mathcal{W} descendants contribute the following factor to the Kac determinant $\det \mathcal{M}_{(N)}$

$$\left((\Lambda + \alpha_+ \rho + \alpha_- \rho^\vee, \alpha_i) - \left(\frac{1}{2}(\alpha_i, \alpha_i) m \alpha_+ + n \alpha_- \right) \right)^{p_\ell(N-mn)} , \quad (6.71)$$

where $m = (\Lambda^{(+)} + \rho, \alpha_i^\vee)$ and $n = (\Lambda^{(-)} + \rho^\vee, \alpha_i)$ are both positive integers. Using Weyl group invariance (6.21), and comparison of the order in Λ of the expression thus obtained with the a priori known order (see *e.g.* [62]), we conclude that the Kac determinant is given, up to a Λ independent factor, by

$$\det \mathcal{M}_N \sim \prod_{\alpha \in \Delta} \prod_{mn \leq N} \left((\Lambda + \alpha_+ \rho + \alpha_- \rho^\vee, \alpha) - \left(\frac{1}{2}(\alpha, \alpha) m \alpha_+ + n \alpha_- \right) \right)^{p_\ell(N-mn)} . \quad (6.72)$$

For $\mathfrak{g} = A_1$ this expression reduces to (2.23) (see also [253,62,326] for various simply-laced cases, and [226] for $\mathfrak{g} = B_2$). The Kac determinant for the supercase $\mathfrak{g} = B(0, n)$ is discussed in [327]).

The Kac determinant can be used to determine the structure of the Verma modules and its quotients. This will be the topic of the next section. Moreover, from the Kac

determinant one can in principle infer what the unitary representations are. Unfortunately, direct analysis of unitarity from the Kac determinant is rather cumbersome and has only been completed in the case of the Virasoro algebra [147,148]. Some results have been obtained however for the \mathcal{W}_N algebras [253,254,255]. Since these results can best be understood in the coset approach to \mathcal{W} -algebras we will postpone this discussion until Chapter 7.

6.4.2. Completely degenerate representations and minimal models

The complete structure of Verma modules (or of any of its quotients) can in principle be determined from the Kac determinant (6.72). Instead of giving a complete catalogue of modules of different types, let us restrict the discussion to the most important class as far as physical applications are concerned. This is the class of irreducible highest weight representations—with highest weight Λ , say—for which the corresponding Verma module M_Λ contains “as many singular vectors as possible,” or is “completely degenerate” [114,109]. More precisely, for which the Verma module M_Λ has ℓ independent singular vectors, one in every simple root direction. From the discussion in Section 6.4.1 it follows that such completely degenerate weights Λ can be parametrized as

$$\Lambda = \alpha_+ \Lambda^{(+)} + \alpha_- \Lambda^{(-)} \quad (6.73)$$

with $\Lambda^{(+)} \in P_+$ and $\Lambda^{(-)} \in P_+^\vee$. The ℓ independent singular vectors occur at levels $(\Lambda^{(+)} + \rho, \alpha_i^\vee)(\Lambda^{(-)} + \rho^\vee, \alpha_i), i = 1, \dots, \ell$. The conformal dimension h_Λ (see (6.32)) can, in this case, be written as

$$h_\Lambda - \frac{c}{24} = -\frac{\ell}{24} + \frac{1}{2} |\alpha_+(\Lambda^{(+)} + \rho) + \alpha_-(\Lambda^{(-)} + \rho^\vee)|^2 . \quad (6.74)$$

If, in addition, α_+^2 is a positive rational number, such that

$$\alpha_+^2 = (k + h^\vee)^{-1} = \frac{p'}{p}, \quad \gcd(p, p') = 1, \quad \gcd(p', r^\vee) = 1, \quad (6.75)$$

it follows from the Kac determinant that the Verma module M_Λ in fact contains an infinite number of singular vectors.

In particular, if we restrict the dominant integral weights $\Lambda^{(+)}, \Lambda^{(-)}$ to the following set

$$\Lambda^{(+)} \in P_+^{p-h^\vee}, \quad \Lambda^{(-)} \in P_+^{\vee p'-h}, \quad (6.76)$$

where, in addition, $p \geq h^\vee, p' \geq h$, we obtain the class of so-called “minimal model representations” (see [114] for A_2 , [109,111] for A_n and D_n and [12] for all \mathfrak{g} simply-laced). It can be argued that the (finite set of) primary fields $\Phi_{\Lambda^{(+)}, \Lambda^{(-)}}(z)$ corresponding to the set (6.76) form a closed operator product algebra, and hence constitute a \mathcal{W} -RCFT.

For these minimal models, the labelling of Λ by the pair $(\Lambda^{(+)}, \Lambda^{(-)})$ is actually slightly redundant. In general, we have to make a further field identification. Because of the relation $p\alpha_+ + p'\alpha_- = 0$, the symmetry (6.21) under the Weyl group of \mathfrak{g} is in fact extended to a symmetry under the affine Weyl group, *i.e.* under

$$\begin{aligned}\Lambda^{(+)} &\rightarrow w(\Lambda^{(+)} + \rho) - \rho + p\beta \\ \Lambda^{(-)} &\rightarrow w(\Lambda^{(-)} + \rho^\vee) - \rho^\vee + p'\beta ,\end{aligned}\tag{6.77}$$

where $w \in W$ and β is an arbitrary element of the long root lattice. The subgroup of transformations (6.77) which leaves invariant the fundamental alcove (6.76) is isomorphic to the center of \mathfrak{g} (*e.g.* \mathbb{Z}_{n+1} for A_n), hence we have to identify fields $\Phi_{(\Lambda^{(+)}, \Lambda^{(-)})}(z)$ that are related by the action of the center of \mathfrak{g} . (In the case when odd integer spin generators in the \mathcal{W} -algebra are present one sometimes has to make a further field identification related to the \mathbb{Z}_2 automorphism $W^{(s_i)}(z) \rightarrow (-1)^{s_i} W^{(s_i)}(z)$.)

For various applications it is useful to have a resolution of an irreducible module L_Λ in terms of free field Fock spaces. One can try to construct such resolutions directly in the \mathcal{W} setting by using the intertwiners (6.70). It is however more convenient to obtain these resolutions from their affine Lie algebra counterparts. This is achieved by applying the functor H_Q to all the terms in a resolution of some irreducible highest weight module of $\widehat{\mathfrak{g}}$ and then making use of (6.30). This functor was studied extensively in [139]. For the case of minimal representations it leads to a conjecture that there exists a resolution of L_Λ with terms

$$C^{(i)} L_\Lambda \cong \bigoplus_{\{w \in \widehat{W} \mid \tilde{\ell}(w)=i\}} \mathcal{F}_{\alpha_+ w * \Lambda^{(+)} + \alpha_- \Lambda^{(-)}} ,\tag{6.78}$$

where $w * \lambda = w(\lambda + \rho) - \rho$ [139,67].

To conclude this section let us specialize some of the above formulae to the minimal model case of the \mathcal{W}_3 algebra (the formulae in the case of \mathcal{W}_2 , *i.e.* the Virasoro algebra, can be found in Section 2.1). The relevant central charge is parametrized by two relatively prime positive integers p and p' ($p, p' \geq 3$) through (see (6.13) and (6.75))

$$c(p, p') = 2 \left(1 - \frac{12(p-p')^2}{pp'} \right) .\tag{6.79}$$

Now, parametrizing $\Lambda^{(+)}$ and $\Lambda^{(-)}$ in (6.76) each by two non-negative integers

$$\begin{aligned}\Lambda^{(+)} &= r_1\Lambda_1 + r_2\Lambda_2, & 0 \leq r_1 + r_2 \leq p - 3, \\ \Lambda^{(-)} &= s_1\Lambda_1 + s_2\Lambda_2, & 0 \leq s_1 + s_2 \leq p' - 3,\end{aligned}\tag{6.80}$$

where $\Lambda_i, i = 1, 2$ are the fundamental weights of $sl(3)$, equation (6.74) yields

$$h^{(2)} \begin{pmatrix} r_1 & s_1 \\ r_2 & s_2 \end{pmatrix} = \frac{(p'(r_i + 1) - p(s_i + 1))G_{ij}(p'(r_j + 1) - p(s_j + 1)) - 2(p - p')^2}{2pp'}, \tag{6.81}$$

where $G_{ij} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ is the inverse Cartan matrix of $sl(3)$. The eigenvalue of the W_3 generator on the highest weight vector of $L_{\Lambda^{(+)}\Lambda^{(-)}}$ follows easily from (6.54)

$$\begin{aligned}h^{(3)} \begin{pmatrix} r_1 & s_1 \\ r_2 & s_2 \end{pmatrix} &= \frac{1}{9pp'} \sqrt{\frac{2}{3(5p - 3p')(5p' - 3p)}} (p'(r_1 - r_2) - p(s_1 - s_2)) \\ &\quad \times (p'(2r_1 + r_2) - p(2s_1 + s_2)) (p'(r_1 + 2r_2) - p(s_1 + 2s_2)).\end{aligned}\tag{6.82}$$

As argued above, the eigenvalues $h^{(i)}$ in (6.81) and (6.82) are invariant under a \mathbb{Z}_3 symmetry. One finds that the generator $S, S^3 = 1$, acts on the labels by

$$S : \begin{pmatrix} r_1 & s_1 \\ r_2 & s_2 \end{pmatrix} \rightarrow \begin{pmatrix} (p - 3) - (r_1 + r_2) & (p' - 3) - (s_1 + s_2) \\ r_1 & s_1 \end{pmatrix}.\tag{6.83}$$

The \mathbb{Z}_2 operation $R, R^2 = 1$, that acts by

$$R : \begin{pmatrix} r_1 & s_1 \\ r_2 & s_2 \end{pmatrix} \rightarrow \begin{pmatrix} r_2 & s_2 \\ r_1 & s_1 \end{pmatrix}, \tag{6.84}$$

leaves $h^{(2)}$ invariant while it changes the sign of $h^{(3)}$. In other words, the eigenvalues $h^{(2)}$ inside the fundamental Kac table (6.80) are six-fold degenerate for $h^{(3)} = 0$ and three-fold degenerate otherwise.

6.4.3. Character formulae

Explicit formulae for the characters of $\mathcal{W}[\widehat{\mathfrak{g}}, k]$ irreducible highest weight modules can be derived straightforwardly from a free field resolution of L_Λ . In this section we want to discuss some explicit formulae for the case of the minimal representations discussed in Section 6.4.2. The required free field resolution, obtained by applying the QDS-functor H_Q to a resolution of an irreducible $\widehat{\mathfrak{g}}$ highest weight module with properly chosen admissible weight, was discussed in Section 6.4.2.

Parametrizing $\Lambda = \alpha_+ \Lambda^{(+)} + \alpha_- \Lambda^{(-)}$, where α_+ , $\Lambda^{(+)}$ and $\Lambda^{(-)}$ are as in (6.75) and (6.76), and using equation (6.74), a simple application of the Euler-Poincaré lemma gives

$$\text{ch}_{L_\Lambda}(q) = \frac{1}{\eta(\tau)^\ell} \sum_{w \in \widehat{W}} \epsilon(w) q^{\frac{1}{2pp'} |p'w(\Lambda^{(+)} + \rho) - p(\Lambda^{(-)} + \rho^\vee)|^2} . \quad (6.85)$$

In the simply-laced case this character formula was first conjectured in [111]. The above derivation was given in [67,139]. From the above construction it is clear that the characters (6.85) can be written as certain residues of characters of $\widehat{\mathfrak{g}}$ irreducible modules with admissible highest weight [139]. This explains an observation made in [261]. This description also, evidently, relates the modular matrix S and the set of modular invariants to that of the admissible Kac-Moody representations.

In Chapter 7 we will see that equation (6.85) coincides, for simply laced Lie algebras \mathfrak{g} , with the branching function associated with a certain coset. This strongly supports the claim of equivalence of the two \mathcal{W} constructions, for which so far (apart from the \mathcal{W}_3 case) no direct proof is available.

7. Coset constructions

7.1. Introduction

The second generic method to obtain examples of \mathcal{W} -algebras from (a pair of) affine Lie algebras, which at first sight is completely unrelated to the QDS reduction scheme of the previous chapter, is through the so-called coset construction [168]. We have already seen a glimpse of it in Section 2.3.2, where we reviewed the affine Sugawara construction $T^{\widehat{\mathfrak{g}}}(z)$ of the Virasoro generator and its GKO generalization $T^{\widehat{\mathfrak{g}}/\widehat{\mathfrak{g}'}}(z)$ to coset pairs. We recall that

$$T^{\widehat{\mathfrak{g}}/\widehat{\mathfrak{g}'}}(z) = T^{\widehat{\mathfrak{g}}}(z) - T^{\widehat{\mathfrak{g}'}}(z), \quad (7.1)$$

with corresponding central charge

$$c(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}'}, k) = c(\widehat{\mathfrak{g}}, k) - c(\widehat{\mathfrak{g}'}, k'). \quad (7.2)$$

The level k' is determined by $k' = jk$ where j is the Dynkin index of the embedding $\mathfrak{g}' \subset \mathfrak{g}$ [102]. Moreover, we have the important property that $T^{\widehat{\mathfrak{g}}/\widehat{\mathfrak{g}'}}(z)$ commutes with the currents of $\widehat{\mathfrak{g}'}$.

The basic idea that generalizes this construction of the coset Virasoro generator is the following.

Suppose we are given a coset pair $(\mathfrak{g}, \mathfrak{g}')$, $\mathfrak{g}' \subset \mathfrak{g}$ of finite-dimensional Lie algebras. Consider their (untwisted) affinizations $\widehat{\mathfrak{g}}' \subset \widehat{\mathfrak{g}}$. Then define the coset algebra $\mathcal{W}_c[\widehat{\mathfrak{g}}/\widehat{\mathfrak{g}'}, k]$ as the set of all normal ordered products of $\widehat{\mathfrak{g}}$ -currents and their derivatives that commute (*i.e.* have regular operator product expansions) with the currents of $\widehat{\mathfrak{g}'}$, modulo the set of null-fields at this particular value of k .²¹ By definition this algebra closes and satisfies the other axioms of a \mathcal{W} -algebra except, possibly, the requirement that it is finitely generated. This is not the case in general, but depends on the specific $\widehat{\mathfrak{g}}$ -modules one is considering as we will see. Another issue is that the coset \mathcal{W} -algebra defined in this way depends crucially on the level k and is in general “exotic” in the terminology of Section 3.1. Varying k will produce \mathcal{W} -algebras of different type. For a specific set of cosets the algebra is however generic. This happens for instance for the so-called diagonal coset pairs $(\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}_{diag})$.

²¹ Since the level k' of $\widehat{\mathfrak{g}'}$ is determined through $k' = jk$ we have suppressed k' in the notation $\mathcal{W}[\widehat{\mathfrak{g}}/\widehat{\mathfrak{g}'}, k]$.

Every coset algebra $\mathcal{W}_c[\widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}', k]$ comes naturally equipped with a set of highest weight modules. Let thereto $L_{\Lambda}^{\widehat{\mathfrak{g}}}$ be an irreducible highest weight module (not necessarily integrable!) of $\widehat{\mathfrak{g}}$ with highest weight Λ and level k . Define $L_{\Lambda, \Lambda'}^{\widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}'}$ as the space of states in $L_{\Lambda}^{\widehat{\mathfrak{g}}}$ which are highest weight under $\widehat{\mathfrak{g}}'$. It is clear that $L_{\Lambda, \Lambda'}^{\widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}'}$ is a highest weight module of the coset algebra $\mathcal{W}_c[\widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}', k]$.²² We have the following decomposition of the $\widehat{\mathfrak{g}}$ -module $L_{\Lambda}^{\widehat{\mathfrak{g}}}$ under the action of $\widehat{\mathfrak{g}}'$

$$L_{\Lambda}^{\widehat{\mathfrak{g}}} = \bigoplus_{\Lambda'} \left(L_{\Lambda, \Lambda'}^{\widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}'} \otimes L_{\Lambda'}^{\widehat{\mathfrak{g}}'} \right), \quad (7.3)$$

where the sum runs over a set of weights Λ' of $\widehat{\mathfrak{g}}'$ at level $k' = jk$ with corresponding irreducible $\widehat{\mathfrak{g}}'$ highest weight modules $L_{\Lambda'}^{\widehat{\mathfrak{g}}'}$. The corresponding equality for the characters reads

$$\text{ch}_{L_{\Lambda}^{\widehat{\mathfrak{g}}}}(q, z) = \sum_{\Lambda'} b_{\Lambda'}^{\Lambda}(q) \text{ch}_{L_{\Lambda'}^{\widehat{\mathfrak{g}}'}}(q, z'), \quad (7.4)$$

where z' is determined as a function of z through the embedding $\mathfrak{g}' \subset \mathfrak{g}$. The branching function $b_{\Lambda'}^{\Lambda}(q)$ is defined through

$$b_{\Lambda'}^{\Lambda}(q) = \text{Tr}_{L_{\Lambda, \Lambda'}} q^{L_0 - c/24}, \quad (7.5)$$

where L_0 and c are the coset expressions given in (7.1) and (7.2).²³

In general the modules $L_{\Lambda, \Lambda'}$ will not be irreducible under the coset algebra $\mathcal{W}_c[\widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}', k]$. The problem of decomposing $L_{\Lambda, \Lambda'}$ into irreducible coset representations has not been completely solved, and will not be discussed in this report. We will only mention that in the case of diagonal coset algebras the modules $L_{\Lambda, \Lambda'}$ are believed to be irreducible. However, one can prove that the modules $L_{0,0}$ are always irreducible [97], implying that $L_{0,0}$ serves as a characteristic Hilbert space for $\mathcal{W}_c[\widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}', k]$. In particular we can study the coset algebras through the character technique (as in Section 6.3.2).

We would also like to remark that by using the character decomposition (7.4) one can derive modular invariant combinations of $\mathcal{W}_c[\widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}', k]$ -characters from those of the affine Lie algebras $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{g}}'$ [60,70,12,82].

In this chapter we will mainly restrict our attention to the diagonal coset algebras $\mathcal{W}_c[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}, (k_1, k_2)]$ for reasons alluded to in the above and also simply because not much is

²² $L_{\Lambda, \Lambda'}^{\widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}'}$ will be infinite-dimensional unless $\widehat{\mathfrak{g}}' \subset \widehat{\mathfrak{g}}$ is conformal (see Section 2.3.2).

²³ In the sequel we will often suppress the superscripts $\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}'$ and $\widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}'$ when no confusion can arise.

known about generic coset models. Our analysis will be most complete in the case when one of the levels, say k_1 , equals one [12], although most of the analysis goes through for generic k_1 . The results for integer $k_1 > 1$ are usually presented in the context of parafermionic algebras which fall outside the scope of this report (see *e.g.* [222,8,158,289,82,314,263,264]).

In the last section we will briefly touch upon the subject of more general coset models (see *e.g.* [97,98,54,172,71,85]).

Before we start discussing diagonal coset models it turns out to be convenient to first discuss a slight generalization of the above construction. The crucial observation is that the Virasoro algebra related to $\widehat{\mathfrak{g}}$ as given by the affine Sugawara construction in fact commutes with the finite-dimensional “horizontal algebra” \mathfrak{g} . We may thus interpret the Sugawara tensor as an element of the coset algebra $\mathcal{W}_c[\widehat{\mathfrak{g}}/\mathfrak{g}, k]$. We will first study this so-called Casimir algebra [140,11,320], and subsequently argue that the diagonal coset algebras can be interpreted as its deformation [12].

One of our main themes in this Chapter will be to establish, for simply-laced \mathfrak{g} , an isomorphism (up to null-fields)

$$\mathcal{W}_c[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}, (1, k_2)] \cong \mathcal{W}_{DS}[\widehat{\mathfrak{g}}, k], \quad (7.6)$$

where k_2 and k are algebraically related. This will be done indirectly by showing that a sufficiently large set of characters (and thus the Kac determinants) of these algebras are identical. It is still an important open problem to prove this isomorphism by direct means and/or to have an a priori understanding why certain coset \mathcal{W} -algebras are equivalent to certain QDS \mathcal{W} -algebras.

7.2. Casimir algebras

7.2.1. Generalities

Casimir algebras were already introduced in Section 3.2. We will briefly recall the construction, and give a reinterpretation in terms of a particular coset. We will indicate that the Casimir algebra is not only a convenient starting point for more general coset constructions, but also provides (at least for a particular c -value) a direct link to the QDS reduction discussed in the last Chapter.

Our starting point is the so-called affine Sugawara construction (see Section 2.3.2). Consider thereto the following composite operator

$$T(z) = \frac{1}{2}\eta^{(2)}(\mathfrak{g}, k) \sum_{a,b} d_{ab}(J^a J^b)(z). \quad (7.7)$$

It is well-defined on highest weight modules of $\widehat{\mathfrak{g}}$ and, for properly chosen normalization constant $\eta^{(2)}(\mathfrak{g}, k) = 1/(k + h^\vee)$ ($k \neq -h^\vee$), satisfies the Virasoro algebra of central charge $c = c(\widehat{\mathfrak{g}}, k) = k \dim \mathfrak{g}/(k + h^\vee)$. In other words, every highest weight module of $\widehat{\mathfrak{g}}$ ($k \neq -h^\vee$) can be extended to a highest weight module of the semi-direct sum of a Virasoro algebra and $\widehat{\mathfrak{g}}$. The currents $J^a(z)$ become primary fields of conformal dimension one with respect to $T(z)$, *i.e.*

$$\underbrace{T(z)J^a(w)} = \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w}, \quad (7.8)$$

which, in particular, implies

$$[J_0^a, T(z)] = 0, \quad (7.9)$$

i.e. $T(z)$ is a singlet under the horizontal (finite-dimensional) subalgebra \mathfrak{g} of $\widehat{\mathfrak{g}}$. We conclude that, in the terminology of Section 7.1, the Sugawara tensor $T(z)$ is an element of the \mathcal{W} -algebra $\mathcal{W}[\widehat{\mathfrak{g}}/\mathfrak{g}, k]$ related to the coset pair $(\widehat{\mathfrak{g}}, \mathfrak{g})$. To construct other singlet fields under \mathfrak{g} is straightforward. Consider thereto a generic field

$$W_{i_1, \dots, i_n}^{b_1, \dots, b_n}(z) = (\partial^{i_1} J^{b_1} \dots \partial^{i_n} J^{b_n})(z). \quad (7.10)$$

Although the full operator product expansion between the currents $J^a(z)$ and $W_{i_1, \dots, i_n}^{b_1, \dots, b_n}(w)$ is hard to evaluate, it is easily seen that the simple pole term only receives contributions from the simple pole term in (2.69). Therefore

$$[J_0^a, W_{i_1, \dots, i_n}^{b_1, \dots, b_n}(z)] = \sum_{i=1}^n \sum_c f^{ab_i}_c W_{i_1, \dots, i_n}^{b_1, \dots, b_{i-1}, c, b_{i+1}, \dots, b_n}(z). \quad (7.11)$$

This implies that the field $W_{i_1, \dots, i_n}(z) = \sum_{b_1, \dots, b_n} d_{b_1 \dots b_n} W_{i_1, \dots, i_n}^{b_1, \dots, b_n}(z)$ commutes with \mathfrak{g} if and only if $d_{b_1 \dots b_n}$ is an invariant tensor under \mathfrak{g} , *i.e.* if and only if $\sum_{b_1, \dots, b_n} d_{b_1 \dots b_n} T^{b_1} \dots T^{b_n}$ is an element in the center of the universal enveloping algebra of \mathfrak{g} , *i.e.* a Casimir operator. For this reason we will also refer to the singlet algebra $\mathcal{W}[\widehat{\mathfrak{g}}/\mathfrak{g}, k]$ as the Casimir algebra of \mathfrak{g} at level k .

Notice that we may assume that the invariant tensors $d_{a_1 \dots a_n}$ are completely symmetric, since for any anti-symmetric pair of indices the field can be reduced by using

$$(J^a J^b)(z) - (J^b J^a)(z) = f^{ab}_c (\partial J^c)(z). \quad (7.12)$$

It is clear that the algebra of all singlet fields $W_{i_1, \dots, i_n}(z)$ closes. To establish that they give rise to a \mathcal{W} -algebra as defined in Section 3.1, one has to find a subset consisting of a

finite number of primary fields which—together with their derivatives and normal ordered products thereof—form a closed operator product algebra. The other conditions for this to be a \mathcal{W} -algebra are automatically fulfilled. The above procedure may (and will) clearly depend on the value of the level k .

The simplest fields²⁴

$$T^{(M)}(z) = \frac{1}{M!} \sum_{a_1, \dots, a_M} d_{a_1 \dots a_M} (J^{a_1} \dots J^{a_M})(z), \quad (7.13)$$

are usually, by abuse of language, referred to as the Casimir operators of $\widehat{\mathfrak{g}}$. It is easily shown that the operators $T^{(M)}(z)$ are primary with respect to (7.7) of dimension M , provided the d -symbols are chosen to be traceless. In general the operators $T^{(M)}(z)$ will not form a complete set of generators of $\mathcal{W}[\widehat{\mathfrak{g}}/\mathfrak{g}, k]$. It can however be argued [262] that they do generate the complete singlet algebra for \mathfrak{g} simply-laced and $k = 1$.

7.2.2. Character technique

In Section 6.3.2 we have seen that the character technique provides a very powerful tool for analysing the set of generators of a \mathcal{W} -algebra. In this section we apply this technique to the Casimir algebras of the previous section.

Characters of the coset algebra $\mathcal{W}[\widehat{\mathfrak{g}}/\mathfrak{g}, k]$ are constructed, according to the general philosophy of Section 7.1, by decomposing the characters of $\widehat{\mathfrak{g}}$ with respect to the characters of \mathfrak{g} . Consider the branching of an integrable highest weight module L_Λ of $\widehat{\mathfrak{g}}$ at level k in terms of irreducible highest weight modules L_λ of \mathfrak{g}

$$\text{ch}_{L_\Lambda}^{\widehat{\mathfrak{g}}}(q, z) = \sum_{\lambda \in P_+} b_\lambda^\Lambda(q) \text{ch}_{L_\lambda}^{\mathfrak{g}}(z). \quad (7.14)$$

For convenience we will also use $\Phi_\lambda^\Lambda(q) \equiv q^{-s_\Lambda} b_\lambda^\Lambda(q)$ where

$$s_\Lambda = \frac{|\Lambda + \rho|^2}{2(k + h^\vee)} - \frac{|\rho|^2}{2h^\vee} = \frac{c_\Lambda}{2(k + h^\vee)} - \frac{1}{24} \frac{k \dim \mathfrak{g}}{k + h^\vee}, \quad (7.15)$$

so that $\Phi_\lambda^\Lambda(q)$ has an expansion in terms of integer powers of q . Explicitly [216],

$$b_\lambda^\Lambda(q) \equiv \text{Tr}_{L_{\Lambda, \lambda}} q^{L_0 - c/24} = \sum_{w \in W} \epsilon(w) c_{w(\lambda + \rho) - \rho + k\Lambda_0}^\Lambda(q) q^{\frac{1}{2k} |w(\lambda + \rho) - \rho|^2}, \quad (7.16)$$

²⁴ In fact, since $d_{a_1 \dots a_M}$ is assumed to be completely symmetric there is no need to normal order this expression. The factor $M!$ is inserted for convenience.

where the $c_\lambda^\Lambda(q)$ are the so-called Kac-Peterson string functions.

We will discuss several examples

(i) \mathfrak{g} simply-laced, $k = 1$.

In this case there is a unique string function $c_{\Lambda_0}^{\Lambda_0}(q) = \left(q^{\frac{1}{24}} \prod(1 - q^n)\right)^{-\ell}$. Applying the denominator formula (9.3) to (7.16) one finds

$$b_\lambda^\Lambda(q) = \frac{q^{\frac{1}{2}|\lambda|^2} \prod_{\alpha \in \Delta_+} (1 - q^{(\lambda+\rho, \alpha)})}{\left(q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n)\right)^\ell}, \quad (7.17)$$

for $\lambda \in P_+ \cap (Q + \Lambda)$ and zero otherwise. In particular, the vacuum character is given by

$$\begin{aligned} \Phi_0^{\Lambda_0}(q) &= \frac{\prod_{\alpha \in \Delta_+} (1 - q^{(\rho, \alpha)})}{\prod_{n \geq 1} (1 - q^n)^\ell} = \frac{\prod_{i=1}^\ell (\prod_{n=1}^{e_i} (1 - q^n))}{\prod_{n \geq 1} (1 - q^n)^\ell} \\ &= \frac{1}{\prod_{i=1}^\ell (F_{e_i+1}(q))}, \end{aligned} \quad (7.18)$$

where $F_s(q)$ is defined in (6.39). Since this is precisely the vacuum character of a \mathcal{W} -algebra of type $\mathcal{W}(e_1 + 1, \dots, e_\ell + 1)$ with independent generators, the above analysis suggests that the singlet algebra $\mathcal{W}_c[\widehat{\mathfrak{g}}/\mathfrak{g}, k = 1]$ is of this type. Exactly the same result (7.18) was obtained in (6.43) for the vacuum character of the algebra $\mathcal{W}_{DS}[\widehat{\mathfrak{g}}, k]$ obtained by the quantum DS-reduction. The main difference, however, is that the analysis in Section 6.3.2 is valid for generic values of k , while the above analysis is restricted to a particular value for k , namely $k = 1$. Another difference is that while the analysis in Section 6.3.2 also applies to non simply-laced Lie algebras, the above analysis is restricted to simply-laced ones. For non simply-laced Lie algebras \mathfrak{g} the singlet algebra is essentially different from the QDS reduction based on \mathfrak{g} . In particular the singlet algebra contains additional fields beyond those corresponding to the Casimirs of \mathfrak{g} . We discuss two examples of this type.

(ii) $\widehat{\mathfrak{g}} = B_\ell^{(1)}$ at level $k = 1$.

Using the explicit formulae for the string functions in [216] one arrives, after a straightforward calculation, at the following formula for the vacuum character of the singlet algebra

$$\Phi_0^{\Lambda_0}(q) = \frac{\prod_{i=1}^\ell \prod_{n=1}^{e_i} (1 - q^n)}{\prod_{n \geq 1} (1 - q^n)^\ell} \times \frac{1}{2} \left(\prod_{n \geq \ell} (1 + q^{n+\frac{1}{2}}) + \prod_{n \geq \ell} (1 - q^{n+\frac{1}{2}}) \right), \quad (7.19)$$

where $\{e_i\} = \{1, 3, 5, \dots, 2\ell - 1\}$ is the set of exponents of B_ℓ . The last term in (7.19) equals the vacuum character of a fermionic dimension $\ell + \frac{1}{2}$ field projected onto the \mathbb{Z}_2 even sector. So, we conclude that the singlet algebra in this case is the bosonic projection (compare with Section 5.3.2) of an algebra of type $\mathcal{W}(2, 4, \dots, 2\ell, \ell + \frac{1}{2})$ (see also [327,330] and [62] for $\ell = 1$). The corresponding QDS reduction is the one based on the affine Lie superalgebra $B(0, \ell)^{(1)}$ (see Section 6.3.3).

(iii) $\widehat{\mathfrak{g}} = G_2^{(1)}$ at level $k = 1$.

A straightforward computation, using the string functions in [216], gives

$$\begin{aligned} \Phi_0^{\Lambda_0}(q) &= (1 - q)(1 - q^2)(1 - q^3) \left((1 + q^2 + q^3)c_{\Lambda_0}^{\Lambda_0} - q^{\frac{1}{3}}(1 + q + q^3)c_{\Lambda_2}^{\Lambda_0} \right) \\ &= \frac{1}{F_2 F_6 F_8 F_{10} F_{12}^2 F_{14}^2 F_{15} F_{16} F_{17} F_{18}^2} (1 - 3q^{22} - 7q^{23} - 14q^{24} + \mathcal{O}(q^{25})), \end{aligned} \tag{7.20}$$

which suggests that the singlet algebra of $G_2^{(1)}$ at $k = 1$ is of type $\mathcal{W}(2, 6, 8, 10, 12_2, 14_2, 15, 16, 17, 18_2)$. So, apart from the Casimir operators of spin 2 and 6, we find other integer spin operators. They can presumably be understood as composites of (among others) a spin $8/3$ operator, whose existence can be anticipated in the level-1 non-simply laced vertex operator realization of $G_2^{(1)}$ [170,39]. In [329] it is argued that, in fact, all generators can be found by taking successive OPE's of the dimension 2 and 6 fields and an additional field of dimension $17/5$. It is not known whether there exists a QDS reduction leading to the same algebra.

7.2.3. 3rd order Casimir example

In this section we present a detailed example. The easiest nontrivial illustration of the above considerations is the algebra of a third order Casimir operator. These exist only for the algebras $A_{N-1} \cong sl(N)$, $N \geq 3$. We have

$$\begin{aligned} d_{ab} &= \text{Tr}(T_a T_b), \\ f_{abc} &= \text{Tr}([T_a, T_b] T_c), \\ d_{abc} &= \text{Tr}(\{T_a, T_b\} T_c). \end{aligned} \tag{7.21}$$

Group indices are raised and lowered by means of the Killing metric d_{ab} (and its inverse d^{ab}). The tensors d_{ab} , d_{abc} are completely symmetric while f_{abc} is completely antisymmetric in its indices. With respect to an antihermitian basis $\{T_a\}$ the tensors d_{ab} , f_{abc} are

real while d_{abc} is pure imaginary. Let us, for future use, list a set of useful contraction identities [242,11]

$$\begin{aligned}
d^{ab}d_{abc} &= 0, \\
f_a{}^{bc}f_{dbc} &= (-2N)d_{ad}, \\
d_a{}^{bc}d_{dbc} &= 2\left(\frac{N^2-4}{N}\right)d_{ad}, \\
f_{ad}{}^bf_{be}{}^cf_{cf}{}^a &= Nf_{def}, \\
d_{ad}{}^bf_{be}{}^cf_{cf}{}^a &= Nd_{def}, \\
d_{ad}{}^bd_{be}{}^cf_{cf}{}^a &= \left(\frac{N^2-4}{N}\right)f_{def}, \\
d_{ad}{}^bd_{be}{}^cd_{cf}{}^a &= \left(\frac{N^2-12}{N}\right)d_{def}.
\end{aligned} \tag{7.22}$$

In addition we have the Jacobi identities

$$\begin{aligned}
f_{ad}{}^ef_{ebc} + f_{bd}{}^ef_{eca} + f_{cd}{}^ef_{eab} &= 0, \\
f_{ad}{}^ed_{ebc} + f_{bd}{}^ed_{eca} + f_{cd}{}^ed_{eab} &= 0, \\
f_{ab}{}^cf_{cde} &= \frac{4}{N}(d_{ae}d_{bd} - d_{ad}d_{be}) + (d_{bd}{}^cd_{cae} - d_{ad}{}^cd_{cbe}),
\end{aligned} \tag{7.23}$$

and a relation valid for $N = 3$ (*i.e.* $sl(3)$) only

$$d_{ab}{}^cd_{cde} = \frac{2}{3}(d_{ad}d_{be} + d_{ae}d_{bd} - d_{ab}d_{de}) - \frac{1}{3}(f_{ad}{}^cf_{cbe} + f_{ae}{}^cf_{cbd}). \tag{7.24}$$

As an intermediate step in the calculation of the OPE $T^{(3)}(z)T^{(3)}(w)$, it is convenient to define

$$T_a^{(2,1)}(z) = \frac{1}{2}d_{abc}(J^bJ^c)(z). \tag{7.25}$$

It is straightforward, using the Wick theorem (2.37), to verify that $T_a^{(2,1)}(z)$ is a primary field of conformal dimension 2. The OPE's of $T_a^{(2,1)}(z)$ and $T^{(3)}(z)$ with the elementary fields $J_a(z)$ are given by

$$\begin{aligned}
J_a(z)T_b^{(2,1)}(w) &= \frac{k + \frac{1}{2}N}{(z-w)^2}d_{ab}{}^cJ_c(w) + \frac{1}{z-w}f_{ab}{}^cT_c^{(2,1)}(w) + \dots, \\
J_a(z)T^{(3)}(w) &= \frac{k + N}{(z-w)^2}T_a^{(2,1)}(w) + \dots.
\end{aligned} \tag{7.26}$$

Here we have made repeated use of the identities (7.22) and (7.23). Note that the $(z-w)^{-1}$ -terms imply that $T_a^{(2,1)}(z)$ and $T^{(3)}(z)$ transform under the adjoint and singlet representation of the horizontal algebra A_{N-1} , respectively.

After a tedious calculation, using rearrangement lemmas such as (2.38), one arrives at the following result

$$\begin{aligned}
W(z)W(w) &= \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} + \frac{1}{(z-w)^2} (2\beta\Lambda + \frac{3}{10}\partial^2 T + R^{(4)})(w) \\
&\quad + \frac{1}{z-w} (\beta\partial\Lambda + \frac{1}{15}\partial^3 T + \frac{1}{2}\partial R^{(4)})(w) + (WW)(w) + \dots
\end{aligned} \tag{7.27}$$

where

$$c = c(A_{N-1}^{(1)}, k) = \frac{k(N^2 - 1)}{k + N}, \tag{7.28}$$

$\Lambda(z)$ and β are defined in equations (3.6), (3.7), respectively, and

$$W(z) = \eta^{(3)}(A_{N-1}, k) T^{(3)}(z), \quad \eta^{(3)}(A_{N-1}, k) = \frac{1}{k + N} \sqrt{\frac{N}{(k + \frac{1}{2}N)(N^2 - 4)}}. \tag{7.29}$$

Apart from the identity field and its descendants, an additional primary field $R^{(4)}(z)$ of dimension 4, and its first descendant $\partial R^{(4)}(z)$, enter in (7.27). Explicitly,

$$R^{(4)}(z) = \left(-2b^2\Lambda - \frac{4}{5}\partial^2 T + \frac{1}{2(k+N)}(\partial J^a \partial J_a) + (k+N) \left(\eta^{(3)} \right)^2 (T^{(2,1)a} T_a^{(2,1)}) \right) (z). \tag{7.30}$$

The term $(T_a^{(2,1)} T^{(2,1)a})(z)$ involves the contraction of two 3rd order d -symbols. For $N \geq 4$ this will produce an independent 4th order d -tensor, corresponding to the 4th order Casimir of $sl(N)$. For $N = 3$, however, no such independent 4th order Casimir exists. Using the explicit d -tensor contraction (7.24) one can show that for $N = 3$

$$R^{(4)}(z) = f(k) \left((TT) - \frac{8+c}{12}\partial^2 T + \frac{22+5c}{24(k+3)}(\partial J^a \partial J_a) \right) (z), \tag{7.31}$$

where

$$f(k) = -\frac{36(k+3)^2}{5(31k+33)(2k+3)}. \tag{7.32}$$

Clearly, if the field $R^{(4)}(z)$ had not been present in (7.27) we would have reproduced exactly the \mathcal{W}_3 algebra of (3.5). So, we are led to the question under which circumstances we can consistently put $R^{(4)}(z)$ equal to zero or, in other words, if there are cases for which $R^{(4)}(z)$ is a null-field. There are various ways of studying this question. One could for instance attempt to show that the field $R^{(4)}(z)$ —together with the fields generated by

taking further OPE's with $R^{(4)}(z)$ —form an ideal in the full operator product algebra. As a first step in this direction we have (for $N = 3$)

$$T^{(3)}(z)R^{(4)}(w) = f(k) \left\{ \frac{3(1 - \frac{1}{2}c)}{(z-w)^4} \left(T^{(3)}(w) + \dots \right) + \frac{1}{(z-w)^2} \left(R^{(5)}(w) + \dots \right) \right\} + \dots, \quad (7.33)$$

where $R^{(5)}(z)$ is yet another primary field (of dimension 5), and the dots stand for descendant fields. From this we learn that only for $c = 2$ (*i.e.* $k = 1$) it is possible for $R^{(4)}(z)$ to be a null-field. One would now have to take OPE's with $R^{(5)}(z)$ and continue, a priori, ad infinitum. However, knowing that $R^{(4)}(z)$ can only decouple for $N = 3, k = 1$ we may try to shortcut this calculation by using a specific realization in which the null-fields vanish automatically.

For simply laced Lie algebras \mathfrak{g} at level one, such a realization is provided by the vertex operator realization [180,27,141,311] in terms of rank $\mathfrak{g} = \ell$ free bosonic scalar fields $\phi^i(z)$. Specifically, in the Cartan-Weyl basis, the generators are realized by

$$\begin{aligned} H^i(z) &= i\partial\phi^i(z), \\ E^\alpha(z) &= c_{-\alpha}e^{i\alpha\cdot\phi(z)}, \end{aligned} \quad (7.34)$$

where the $c_{-\alpha}$ are certain cocycle factors whose specific form is irrelevant for the present discussion (see [171], and references therein, for more details). The Fock space \mathcal{F}_Λ^ϕ of the scalar fields $\phi^i(z), i = 1, \dots, \ell$ is irreducible, hence isomorphic to L_Λ (see *e.g.* [171] and references therein). Inserting the realization (7.34) in (7.13) one finds (see [11] for more details)

$$\begin{aligned} T(z) &= \frac{1}{2}(H^1H^1 + H^2H^2), \\ W(z) &= -\frac{1}{6}(3H^1H^1H^2 - H^2H^2H^2), \end{aligned} \quad (7.35)$$

while $R^{(4)}(z)$ vanishes upon insertion of (7.34).²⁵ Alternatively, one could compute the OPA of $W(z)$ directly from (7.35), in which case one would obtain (7.27) (for $c = 2$) without the field $R^{(4)}(z)$ being present.

Yet another method for showing that $R^{(4)}(z)$ is a null-field for $N = 3, k = 1$ is to make use of the character technique of Section 7.2.2. For $sl(3), k = 1$ equation (7.18) gives

$$\Phi_0^{\Lambda_0}(q) = 1 + q^2 + 2q^3 + 3q^4 + 3q^5 + \dots \quad (7.36)$$

²⁵ In fact one can write $R^{(4)}(z) \sim (J_a R^{(3)a})(z)$ for some primary field $R^{(3)a}(z)$ (see Section 7.3.2), and it is even easier to show the vanishing of $R^{(3)a}(z)$.

In particular there should be three independent states at energy level four. By calculating the determinant of the inner product matrix of the three states $\{L_{-2}L_{-2}|0\rangle, L_{-4}|0\rangle, W_{-4}|0\rangle\}$ it is straightforward to show that there are no nullstates contained in this set. This also proves that $R^{(4)}(z)$ has to be null.

At this point one can make a remarkable observation that if one considers the expression (7.13) ($M = 2, 3$) where the sum over the indices is restricted to the CSA of $sl(3)$, the result is, in fact, proportional to (7.35). This is of course well-known for the Virasoro generator (“quantum equivalence” [171]), where it is a consequence of the fact that for simply laced \mathfrak{g} at level $k = 1$ the embedding $u(1)^\ell \subset \mathfrak{g}$ is conformal. The expressions (7.13) restricted to the CSA are precisely the Weyl invariant polynomials on \mathfrak{h} .

In Section 6.3.2 we have already argued that the \mathcal{W} -algebra $\mathcal{W}_{DS}[\widehat{\mathfrak{g}}, k]$ at the specific point $\alpha_\pm = \pm 1$ is in fact isomorphic to $\mathcal{W}_c[\widehat{\mathfrak{g}}/\mathfrak{g}, 1]$ for \mathfrak{g} simply-laced. We can now easily verify that the equations (7.35) and (6.55) are, in fact, form identical. The above quantum equivalence now also explains, in concrete terms, why there exists the Weyl invariance (6.21) (at the point $\alpha_\pm = \pm 1$).

At level $k = -N$ the fields $T^{(2)}(z)$ and $T^{(3)}(z)$ commute with the affine currents (see *e.g.* (7.26)) and thus generate an abelian algebra. More generally, one can prove that $\mathcal{W}(\widehat{\mathfrak{g}}/\mathfrak{g}, -\mathfrak{h}^\vee)$ is isomorphic to the center of the universal enveloping algebra of $\widehat{\mathfrak{g}}$ at $k = -\mathfrak{h}^\vee$ [119]. This can be used—and in fact was the original motivation in the mathematics literature for studying \mathcal{W} -algebras—to prove the Kac-Kazhdan conjecture [215,183,175,244].

7.3. $G \times G/G$ coset conformal field theories

7.3.1. Deforming the singlet algebra

In the previous section we have examined the coset algebra $\mathcal{W}_c[\widehat{\mathfrak{g}}/\mathfrak{g}, k]$ and, for simply-laced $\widehat{\mathfrak{g}}$ at the specific value $k = 1$, established the equivalence with the QDS algebra $\mathcal{W}_{DS}[\widehat{\mathfrak{g}}, k]$. The latter algebra was however shown to exist for a continuum of c -values so the question that arises naturally is whether there exists a coset algebra that corresponds to the generic case. In this section we will argue that the relevant coset algebra is $\mathcal{W}_c[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}, (1, k_2)]$. In fact, we will establish that the coset algebra $\mathcal{W}_c[\widehat{\mathfrak{g}}/\mathfrak{g}, k_1]$ can be interpreted as the $k_2 \rightarrow \infty$ limit of the coset algebra $\mathcal{W}_c[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}, (k_1, k_2)]$. In the subsequent section we will elaborate on this by explicit construction of the spin-3 generator of $\mathcal{W}_c[A_{N-1}^{(1)} \oplus$

$A_{N-1}^{(1)}/A_{N-1}^{(1)}, (1, k_2)]$ as a deformation of that of $\mathcal{W}_c[A_{N-1}^{(1)}/A_{N-1}, 1]$. This will generalize the GKO construction (7.1) of the coset Virasoro algebra.

Let us distinguish the two algebras in $\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}$ by subscripts, *i.e.* $\widehat{\mathfrak{g}}_{(1)} \oplus \widehat{\mathfrak{g}}_{(2)}$, and denote the respective generators by $J_{(1)}^a(z)$ and $J_{(2)}^a(z)$. The diagonal subalgebra is generated by $J_{diag}^a(z) = J_{(1)}^a(z) + J_{(2)}^a(z)$.

Let $\mathcal{H}_{(i)}$ be the characteristic Hilbert space of $\widehat{\mathfrak{g}}_{(i)}$ and let $|\phi\rangle \in \mathcal{H}_{(1)} \otimes \mathcal{H}_{(2)}$ be a singlet under $\widehat{\mathfrak{g}}_{diag}$, then

$$\left(J_{(1)n}^a + J_{(2)n}^a \right) |\phi\rangle = 0, \quad \forall n \geq 0. \quad (7.37)$$

Denote by P_N the orthogonal projection onto the eigenspace $L_0(\widehat{\mathfrak{g}}_{(2)}) = N$. We have

$$|\phi\rangle = \sum_{0 \leq N \leq h_\phi} P_N |\phi\rangle. \quad (7.38)$$

By applying P_N to (7.37) one can argue (see [71] for details) that the map $|\phi\rangle \rightarrow P_0 |\phi\rangle$ is injective, and that moreover $J_{(1)0}^a P_0 |\phi\rangle = 0$. We thus have an injective map

$$P_0 : \mathcal{W}[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}, (k_1, k_2)] \rightarrow \mathcal{W}[\widehat{\mathfrak{g}}/\mathfrak{g}, k_1]. \quad (7.39)$$

By rescaling $J_{(2)n}^a$ by $\sqrt{k_2}$ one can furthermore argue that in fact $|\phi\rangle \rightarrow P_0 |\phi\rangle$ in the limit $k_2 \rightarrow \infty$.

To establish that the map P_0 in (7.39) is actually an isomorphism in the limit $k_2 \rightarrow \infty$ we consider the branching function (see *e.g.* [217,219])

$$b_{\Lambda_3}^{\Lambda_1, \Lambda_2}(q) = \sum_{w \in \widehat{W}} \epsilon(w) c_{\Lambda_3 - w * \Lambda_2}^{\Lambda_1}(q) q^{\frac{|w(\Lambda_2 + \rho)(k_1 + k_2 + h^\vee) - (\Lambda_3 + \rho)(k_2 + h^\vee)|^2}{2k_1(k_2 + h^\vee)(k_1 + k_2 + h^\vee)}}. \quad (7.40)$$

Here Λ_1, Λ_2 are integrable weights at levels k_1 and k_2 , respectively, and Λ_3 is an integrable weight at level $k_1 + k_2$ such that $\Lambda_1 + \Lambda_2 - \Lambda_3 \in Q$, where Q is the long root lattice of \mathfrak{g} .

In the limit $k_2 \rightarrow \infty$ only the elements $w \in W$ in the sum (7.40) will contribute since the others are of $\mathcal{O}(q^{k_2})$. We thus have

$$b_{\Lambda_3}^{\Lambda_1, \Lambda_2}(q) \xrightarrow{k_2 \rightarrow \infty} \sum_{w \in W} \epsilon(w) c_{\Lambda_3 - w * \Lambda_2}^{\Lambda_1}(q) q^{\frac{|w * \Lambda_2 - \Lambda_3|^2}{2k_1}}. \quad (7.41)$$

In particular, for the vacuum character $\Lambda_1 = k_1 \Lambda_0, \Lambda_2 = k_2 \Lambda_0, \Lambda_3 = (k_1 + k_2) \Lambda_0$, this equation reduces to the expression (7.16) for $\Lambda = k \Lambda_0, \lambda = 0$, *i.e.* the vacuum character

for the coset pair $(\widehat{\mathfrak{g}}, \mathfrak{g}, k_1)$. This implies that the map (7.41) becomes surjective in the limit $k_2 \rightarrow \infty$, and that for large enough k_2 the generators of $\mathcal{W}[\widehat{\mathfrak{g}}/\mathfrak{g}, k_1]$ can be deformed to elements of $\mathcal{W}[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}, (k_1, k_2)]$ (see also [328]).

The above results imply that the generators of $\mathcal{W}[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}, (1, k_2)]$ are independent in a neighbourhood of $k_2 = \infty$, *i.e.* for k_2 large enough, and thus have to form a closed algebra for a continuum of k_2 values and hence, because the operator expansion coefficients are algebraic in k_2 , for all k_2 values except for those for which the OPE coefficients develop singularities [328].

7.3.2. Concrete example

In this section we show, by explicit construction, that the Casimir generator $T^{(3)}(z)$ of $\mathcal{W}[A_{N-1}^{(1)}/A_{N-1}, k_1]$ (see (7.13)) can be deformed (for generic values of (k_1, k_2)) to an element in $\mathcal{W}[A_{N-1}^{(1)} \oplus A_{N-1}^{(1)}/A_{N-1}^{(1)}, (k_1, k_2)]$ as we have argued in the previous section. We will also determine the algebra of this dimension-three operator.

We introduce the following mixed contractions

$$\widetilde{T}^{(i,j)}(z) = \frac{1}{i!j!} \sum_{a_1, \dots, a_3} d_{a_1 \dots a_i a_{i+1} \dots a_3} \left(J_{(1)}^{a_1} \dots J_{(1)}^{a_i} J_{(2)}^{a_{i+1}} \dots J_{(2)}^{a_3} \right) (z), \quad (7.42)$$

and take the following ansatz for the coset generator

$$\widetilde{T}^{(3)}(z) = \sum_{i+j=3} a^{(i,j)}(A_{N-1}, k_1, k_2) \widetilde{T}^{(i,j)}(z). \quad (7.43)$$

It is easy to check that the requirement that (7.43) is a singlet under the diagonal $\widehat{\mathfrak{g}}$, *i.e.* that the OPE with the currents $\widetilde{J}^a = J_{(1)}^a + J_{(2)}^a$ is regular, uniquely fixes the coefficients $a^{(i,j)}(A_{N-1}, k_1, k_2)$ up to an overall multiplicative constant

$$a^{(i,j)}(A_{N-1}, k_1, k_2) = (-1)^j \prod_{p=i+1}^3 \left(k_1 + (p-1) \frac{N}{2} \right) \prod_{q=j+1}^3 \left(k_2 + (q-1) \frac{N}{2} \right). \quad (7.44)$$

The coefficient $\widetilde{\eta}^{(3)}(A_{N-1}, k_1, k_2)$ needed to bring the field $\widetilde{W}^{(3)}(z) = \widetilde{\eta}^{(3)} \widetilde{T}^{(3)}(z)$ to its standard normalization $\langle \widetilde{W}^{(3)} | \widetilde{W}^{(3)} \rangle = \widetilde{c}/3$, where

$$\begin{aligned} \widetilde{c} &= c(A_{N-1}^{(1)} \oplus A_{N-1}^{(1)}, A_{N-1}^{(1)}, (k_1, k_2)) = c(A_{N-1}^{(1)}, k_1) + c(A_{N-1}^{(1)}, k_2) - c(A_{N-1}^{(1)}, k_1 + k_2) \\ &= \frac{k_1 k_2 (k_1 + k_2 + 2N) (N^2 - 1)}{(k_1 + N)(k_2 + N)(k_1 + k_2 + N)}, \end{aligned} \quad (7.45)$$

is given by

$$\begin{aligned} \left(\tilde{\eta}^{(3)}\right)^{-2} &= \frac{N^2 - 4}{N} (k_1 + N)^2 (k_2 + N)^2 (k_1 + k_2 + N)^2 \\ &\quad \left(k_1 + \frac{N}{2}\right) \left(k_2 + \frac{N}{2}\right) \left(k_1 + k_2 + \frac{3N}{2}\right). \end{aligned} \quad (7.46)$$

The results (7.43) – (7.46) were first given in [12]. The field $\widetilde{W}^{(3)}(z)$ is primary of dimension 3 under the coset Virasoro generator $\widetilde{T}(z)$.

It is now a straightforward exercise to check that by taking the limit $k_2 \rightarrow \infty$ in (7.43) – (7.46) one recovers the expression (7.29) for the spin-3 Casimir operator. This illustrates the claim of Section 7.3.1 that every generator in $\mathcal{W}[\widehat{\mathfrak{g}}/\mathfrak{g}, k_1]$ can be deformed (with the possible exception of a finite number of (k_1, k_2) -values) to an element of $\mathcal{W}[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}, (k_1, k_2)]$.

By taking the OPE of the generators $\widetilde{W}^{(3)}$ one recovers the result (7.27), in terms of the (tilded) coset fields (7.43) and central charge \tilde{c} given by (7.45). The field $\widetilde{R}^{(4)}(z)$ is a (coset) primary of dimension 4. It is given by an involved expression in terms of the currents $J_{(1)}^a(z)$ and $J_{(2)}^a(z)$ which however drastically simplifies in the case of $N = 3$ and one of the levels (say k_1) is put equal to one. One finds in this case (see [12])

$$\widetilde{R}^{(4)}(z) = c_1(k_2)(R_{(1)ab}^{(2)}R_{(2)}^{(2)ab})(z) + c_2(k_2)(J_{(1)a}R_{(1)}^{(3)a})(z) + c_3(k_2)(J_{(2)a}R_{(1)}^{(3)a})(z), \quad (7.47)$$

where

$$\begin{aligned} R_a^{(3)}(z) &= 3(J_a T)(z) + f_{abc}(J^b \partial J^c)(z) - 2\partial^2 J_a(z), \\ R_{ab}^{(2)}(z) &= (3d_{abd}d_{cd} + 24d_{ac}d_{bd} + 8f_{ace}f_{bd}^e)(J^c J^d)(z), \end{aligned} \quad (7.48)$$

are primary fields of dimension 3 and 2, transforming in the $\underline{8}$ and $\underline{27}$ of the horizontal $sl(3)$, respectively. The coefficients $c_i(k_2)$ have the property that $c_i(1) = 0$. By the same methods as employed in Section 7.2.3, *i.e.* by using the explicit vertex operator construction or by studying the character, one now shows that both $R^{(3)a}(z)$ and $R^{(2)ab}(z)$ are vanishing on the integrable (irreducible) highest weight modules L_Λ at level one. This establishes the existence of a \mathcal{W}_3 -algebra on the coset modules constructed from the $A_2^{(1)} \oplus A_2^{(1)}$ modules $L_\Lambda \otimes M$ at level $(k_1, k_2) = (1, k_2)$ where M is an *arbitrary* $A_2^{(1)}$ -module with highest weight.

We finish this section by briefly mentioning the possibility of ‘supersymmetrizing’ specific coset algebras $\mathcal{W}_c[\widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}', k]$. By this, we mean adding halfinteger dimension operators, constructed out of the $\widehat{\mathfrak{g}}$ currents *and* $\widehat{\mathfrak{g}}$ primary fields, commuting with $\widehat{\mathfrak{g}}'$ and such that the combined set again forms a closed algebra.

The standard example occurs in the context of diagonal cosets $\mathcal{W}_c[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}, (k_1, k_2)]$, \mathfrak{g} simply-laced, by taking $k_1 = \mathfrak{h}^\vee$. It is easily checked, using (2.74), that the adjoint representation at level \mathfrak{h}^\vee has conformal dimension $\frac{1}{2}$. The corresponding highest weight fields $\psi^a(z)$ thus correspond to a set of $\dim(\mathfrak{g})$ free fermions. Let us normalize them such that

$$J_{(1)}^a(z)\psi^b(z) = f^{ab} \frac{\psi^c(w)}{z-w}, \quad \psi^a(z)\psi^b(w) = \frac{d^{ab}}{z-w}. \quad (7.49)$$

Then, a dimension 3/2 operator that commutes with $\widehat{\mathfrak{g}}_{diag}$ is explicitly given by [97,13,172]

$$G(z) = \frac{1}{3}(J_{(1)a}\psi^a)(z) - \frac{\mathfrak{h}^\vee}{k_2}(J_{(2)a}\psi^a)(z) \quad (7.50)$$

For $\mathfrak{g} = su(2)$ this leads to a well-known coset construction of the $N = 1$ superconformal algebra [169]. For higher rank algebras there will, in general, be additional generators of half-integer dimension. The case $\mathfrak{g} = su(3)$, in particular, leads to the super- \mathcal{W}_3 algebra discussed in Section 4.2 [191,4,302]. More general supersymmetrizable coset algebras have been discussed in [71].

7.3.3. Representation theory

Since the coset algebras $\mathcal{W}_c[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}, (k_1, k_2)]$ come naturally equipped with a set of characters (*i.e.* branching functions) $b_{\Lambda_3}^{\Lambda_1, \Lambda_2}(q)$ (see Section 7.1) we can, in principle, study their representation theory by examining the set of branching functions.

Since our main goal will be to argue, for simply-laced \mathfrak{g} , the equivalence of the coset algebra $\mathcal{W}_c[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}, (1, k_2)]$ to the QDS algebra $\mathcal{W}_{DS}[\widehat{\mathfrak{g}}, k]$ we will restrict the following discussion to the case $k_1 = 1$ and \mathfrak{g} simply-laced. The general situation can be treated analogously.

The branching functions $b_{\Lambda_3}^{\Lambda_1, \Lambda_2}(q)$ for the case of integrable weights were already presented in (7.40). We present here a slight generalization [217,219]. Let $\widehat{\Lambda}_1 \in P_+^1$, and suppose $k_2 + \mathfrak{h}^\vee = p/(p' - p) \equiv p/u$. To $\widehat{\Lambda}^{(+)} \in P_+^{p - \mathfrak{h}^\vee}$ we associate the principal admissible weight $\widehat{\Lambda}_2 = \widehat{\Lambda}^{(+)} - (u - 1)(k_2 + \mathfrak{h}^\vee)\Lambda_0$ of level k_2 . Then the branching function for the occurrence of L_{Λ_3} , $\widehat{\Lambda}_3 = \widehat{\Lambda}^{(-)} - (u - 1)(k_2 + \mathfrak{h}^\vee + 1)\Lambda_0$, $\widehat{\Lambda}^{(-)} \in P_+^{p' - \mathfrak{h}^\vee}$ in the decomposition of $L_{\Lambda_1} \otimes L_{\Lambda_2}$ under $\widehat{\mathfrak{g}}_{diag}$ is given by

$$b_{\Lambda_3}^{\Lambda_1, \Lambda_2}(q) = \frac{1}{\eta(\tau)^\ell} \sum_{w \in \widehat{W}} \epsilon(w) q^{\frac{1}{2pp'} |p'w(\Lambda^{(+)} + \rho) - p(\Lambda^{(-)} + \rho)|^2}. \quad (7.51)$$

provided $\Lambda_1 + \Lambda_2 - \Lambda_3 \in Q$ (and vanishing otherwise). In particular, for $p' = p + 1$ we recover (7.40). Now observe that, in fact, (7.51) is identical to (6.85), provided we identify

$$\frac{p' - p}{p} = \frac{1}{k_2 + h^\vee} = \frac{1}{k + h^\vee} - 1. \quad (7.52)$$

This finally proves that $\mathcal{W}_c[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}, (1, k_2)] \cong \mathcal{W}_{DS}[\widehat{\mathfrak{g}}, k]$ (up to null-fields) provided we identify k_2 and k through (7.52). In particular, we read off from (7.51) that we have a set of completely degenerate highest weight modules (*i.e.* nullvectors in all simple root directions) of $\mathcal{W}_c[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}, (1, k_2)]$ labelled by two integrable weights $\Lambda^{(+)} \in P_+^{p-h^\vee}$, $\Lambda^{(-)} \in P_+^{p'-h^\vee}$ of conformal dimension $h_{\Lambda^{(+)}\Lambda^{(-)}}$ given by (see also (6.74))²⁶

$$h_{\Lambda^{(+)}\Lambda^{(-)}} - \frac{c}{24} = -\frac{\ell}{24} + \frac{1}{2pp'} |p'(\Lambda^{(+)} + \rho) - p(\Lambda^{(-)} + \rho)|^2, \quad (7.53)$$

where, using (7.45),

$$c = \frac{k_2(k_2 + 1 + 2h^\vee) \dim \mathfrak{g}}{(1 + h^\vee)(k_2 + h^\vee)(k_2 + 1 + h^\vee)} = \ell \left(1 - \frac{h^\vee(h^\vee + 1)(p - p')^2}{pp'} \right). \quad (7.54)$$

This is, of course, in agreement with (6.13), for $\alpha_+^2 = p'/p$.

Having established the equivalence of the coset and QDS \mathcal{W} -algebra we now want to make some comments on the set of unitary representations for this algebra as promised in Section 6.4.1. Clearly, the realization of $\mathcal{W}_c[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}, (k_1, k_2)]$ on $L_{\Lambda_1 \otimes \Lambda_2, \Lambda_3}$ will be unitary when the $\widehat{\mathfrak{g}}$ modules L_Λ are unitary themselves, *i.e.* when they are integrable. This means $p = k_2 + h^\vee = p' - 1$, $\Lambda^{(+)} \in P_+^{p-h^\vee}$, $\Lambda^{(-)} \in P_+^{p'+1-h^\vee}$, $p \geq h^\vee$ in (7.53), (7.54). The conjecture is that these are *all* unitary modules in the range $c < \ell$, and furthermore that for $c \geq \ell$ there is no restriction on $(\Lambda^{(+)}, \Lambda^{(-)})$ from unitarity. This conjecture is known to be true for $\mathfrak{g} = sl(2)$ [146,147,148,169], and also not in contradiction with the analysis in [253,254].

Modular invariant combinations of the \mathcal{W} -characters $b_{\Lambda_3}^{\Lambda_1, \Lambda_2}(q)$ (7.51) can be constructed out of those for the $\widehat{\mathfrak{g}}$ -characters by using the decomposition (7.4). The simplest example, for \mathcal{W}_3 , is the partition function of the 3-state Potts model which has been discussed in Section 4.2.

7.3.4. Kac determinant

²⁶ We suppress the dependence on $\Lambda_1 \in P_+^1$ since Λ_1 adjusts itself in such a way that $\Lambda_1 + \Lambda_2 - \Lambda_3 \in Q$.

In this section we will derive the Kac determinant (see Section 6.3.1) for the coset \mathcal{W} -algebra $\mathcal{W}_c[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}, (1, k)]$, purely from the coset data, *i.e.* without presupposing a relation to a \mathcal{W} -algebra arising from the QDS reduction. For simplicity we will restrict our attention to the case of simply-laced \mathfrak{g} .

We recall the branching function (7.51)

$$\text{ch}_{L_{\Lambda^{(+)}\Lambda^{(-)}}}(q) = b_{\Lambda_3}^{\Lambda_1, \Lambda_2}(q) = \frac{1}{\eta(\tau)^\ell} \sum_{w \in \widehat{W}} \epsilon(w) q^{\frac{1}{2pp'} |p'w(\Lambda^{(+)} + \rho) - p(\Lambda^{(-)} + \rho)|^2}. \quad (7.55)$$

To derive the Kac determinant we exploit the fact that the branching function (7.55) contains sufficient information on the existence and whereabouts of singular vectors in the $\mathcal{W}[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}]$ Verma modules. To this end we expand the expression given in (7.55) in powers of q in the limit $k \rightarrow \infty$

$$\text{ch}_{L_{\Lambda^{(+)}\Lambda^{(-)}}} = \frac{q^{h_{\Lambda^{(+)}\Lambda^{(-)}} - c/24}}{\prod (1 - q^n)^\ell} \left(1 - \sum_i q^{(\Lambda^{(+)} + \rho, \alpha_i)(\Lambda^{(-)} + \rho, \alpha_i)} + \dots \right), \quad (7.56)$$

where we have depicted only the leading order correction terms in the limit $k \rightarrow \infty$, which arise from the Weyl group elements $w = r_i, i = 1, \dots, \ell$. From (7.56) we conclude that, for k sufficiently large, there exists an $i \in \{1, \dots, \ell\}$ such that the module $L_{\Lambda^{(+)}\Lambda^{(-)}}$ has a singular vector at degree $(\Lambda^{(+)} + \rho, \alpha_i)(\Lambda^{(-)} + \rho, \alpha_i)$. When we parametrize, as usual, $\Lambda = \alpha_+ \Lambda^{(+)} + \alpha_- \Lambda^{(-)}$ the above reasoning shows that $\det \mathcal{M}_N$ will have a vanishing line

$$(\Lambda + \alpha_0 \rho, \alpha_i) - (m\alpha_+ + n\alpha_-), \quad (7.57)$$

for all $m, n \in \mathbb{Z}_+$ such that $N = mn$. Including the contribution of descendant states and using Weyl invariance we conclude, as in Section 6.3.1, that the Kac determinant is given by

$$\det \mathcal{M}_N \sim \prod_{\alpha \in \Delta} \prod_{mn \leq N} ((\Lambda + \alpha_0 \rho, \alpha) - (m\alpha_+ + n\alpha_-))^{p_\ell(N - mn)}. \quad (7.58)$$

This indeed agrees with (6.72) for simply-laced $\widehat{\mathfrak{g}}$.

7.3.5. The limiting \mathcal{W} -algebra of diagonal coset models

We have seen that to every diagonal coset pair $(\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}_{diag})$ one can associate a \mathcal{W} -algebra $\mathcal{W}[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}, (k_1, k_2)]$. This \mathcal{W} -algebra “stabilizes” for fixed k_1 in the limit $k_2 \rightarrow \infty$, where it reduces to the “Casimir algebra” of $\widehat{\mathfrak{g}}$, *i.e.* the algebra $\mathcal{W}[\widehat{\mathfrak{g}}/\mathfrak{g}, k_1]$. One might

pose the question whether this Casimir algebra in turn “stabilizes” in the limit $k_1 \rightarrow \infty$. This appears to be the case. To be able to consider the limiting expression for $b_0^{k\Lambda_0}(q)$ for $k \rightarrow \infty$ in (7.16), let us recall the following explicit expression for the Kac-Peterson string functions (see *e.g.* [66])

$$c_\lambda^\Lambda(q) = \frac{q^{-c/24}}{\prod(1-q^n)^{\dim \mathfrak{g}}} \sum_{w \in \widehat{W}} \epsilon(w) q^{h_{w*\Lambda, \lambda}} \sum_{\{n_\beta \in \mathbb{Z} \mid \sum_{\beta \in \Delta_+} n_\beta \beta = w*\Lambda - \lambda\}} \left(\prod_{\beta \in \Delta_+} \phi_{n_\beta}(q) \right), \quad (7.59)$$

where we have introduced

$$\phi_n = \sum_{m \geq 0} (-1)^m q^{\frac{1}{2}m(m+1) + nm}, \quad \phi_{-n}(q) = q^n \phi_n(q), \quad (7.60)$$

and

$$h_{\Lambda, \lambda} = \frac{(\Lambda, \Lambda + 2\rho)}{2(k + h^\vee)} - \frac{(\lambda, \lambda)}{2k}. \quad (7.61)$$

In the limit $k \rightarrow \infty$ only the finite part of the Weyl group contributes in the sum (7.59). This inserted in (7.16) produces the required limit. As an example consider $\mathfrak{g} = A_1$. We have

$$c_{l\Lambda_1 + (k-l)\Lambda_0}^{k\Lambda_0}(q) \xrightarrow{k \rightarrow \infty} \frac{\phi_{-l/2}(q) - \phi_{-1-l/2}(q)}{\prod(1-q^n)^3}. \quad (7.62)$$

Hence

$$\begin{aligned} \Phi_0^{k\Lambda_0} &\xrightarrow{k \rightarrow \infty} \frac{\phi_0(q) - 2q\phi_1(q) + q^2\phi_2(q)}{\prod(1-q^n)^3} \\ &= \frac{1}{F_2 F_4 F_6^2 F_8^2 F_9 F_{10}^2 F_{12}} (1 - q^{13} - 3q^{14} - 7q^{15} + \mathcal{O}(q^{16})). \end{aligned} \quad (7.63)$$

This suggests the existence of a “limiting \mathcal{W} -algebra” of type $\mathcal{W}(2, 4, 6_2, 8_2, 9, 10_2, 12)$ (see also [56]), which could serve as a “universal \mathcal{W} -algebra” in the sense that it contains all the A_1 coset algebras at finite k -values. In fact, the limiting field content is already reached for a low value of k , namely $k = 6$.²⁷

For $\mathfrak{g} = A_2$ one finds similarly a limiting \mathcal{W} -algebra of type $\mathcal{W}(2, 3, 4, 5, 6_4, 7_2, 8_7, 9_9, 10_{12}, 11_{16}, 12_{26}, 13_{26}, 14_{33}, 15_{33}, 16_{12})$. The limiting field content is reached at $k = 8$ [56].

The relevance of the above considerations remains to be investigated.

²⁷ For $k = 1, \dots, 5$ the \mathcal{W} -algebras are of type $\mathcal{W}(2)$, $\mathcal{W}(2, 4, 6)$, $\mathcal{W}(2, 4, 6_2, 8, 9)$, $\mathcal{W}(2, 4, 6_2, 8_2, 9, 10)$, $\mathcal{W}(2, 4, 6_2, 8_2, 9, 10_2)$, respectively.

7.4. Other cosets

7.4.1. $G \times G'/G'$ coset conformal field theories

The most straightforward generalization of the $\mathcal{W}_c[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}]$ construction of Section 7.3 is the construction of $\mathcal{W}_c[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}'/\widehat{\mathfrak{g}}', (k_1, k_2)]$, where $\mathfrak{g}' \subset \mathfrak{g}$ is an embedding of Dynkin index j . The diagonal subalgebra $\widehat{\mathfrak{g}}'_{diag} \subset \widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}'$ will thus have level $jk_1 + k_2$. Most of the results of Section 7.3 can easily be carried over to this case. In particular, if $|\phi\rangle$ is a $\widehat{\mathfrak{g}}'_{diag}$ singlet in the characteristic Hilbert space of $\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}'$ and $P_N|\phi\rangle$ denotes its orthogonal projection onto the eigenspace $L_0(\widehat{\mathfrak{g}}') = N$ then $P_0|\phi\rangle$ is a \mathfrak{g}' -singlet. The corresponding map

$$P_0 : \mathcal{W}_c[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}'/\widehat{\mathfrak{g}}', (k_1, k_2)] \rightarrow \mathcal{W}_c[\widehat{\mathfrak{g}}/\mathfrak{g}', k_1], \quad (7.64)$$

is injective and turns into an isomorphism in the limit $k_2 \rightarrow \infty$ [71].

The most interesting case occurs when $\widehat{\mathfrak{g}}' \subset \widehat{\mathfrak{g}}$ is conformal (which requires at least that $k_1 = 1$) [71]. Of course, since a conformal coset $\widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}'$ is ‘trivial’, *i.e.* has finite branching rules, one can interpret any $\mathcal{W}_c[\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}'/\widehat{\mathfrak{g}}']$ model as a particular case of a $\mathcal{W}_c[\widehat{\mathfrak{g}}' \oplus \widehat{\mathfrak{g}}'/\widehat{\mathfrak{g}}']$ model. Detailed examples of such coset algebras are discussed in [172,71]. For instance, by taking the conformal embedding $so(N) \subset su(N)$, $N \geq 3$ of Dynkin index $j = 2$,²⁸ one obtains a class of \mathcal{W} -algebras $\mathcal{W}_c[su(N) \oplus so(N)/so(N), (1, m)]$ that are related to a class of parafermions with dihedral D_N -symmetry [113] (see also the remarks in Section 5.3.3). In particular, for $N = 3$, where we have $D_3 \cong S_3$, we obtain the Fateev-Zamolodchikov S_3 (or spin-4/3) parafermionic model [115,8,222,9,290,289]. Let us discuss this particular example in somewhat more detail. That is, we consider the algebra $\mathcal{W}_c[A_2^{(1)} \oplus A_1^{(1)}/A_1^{(1)}, (1, m)]$, where $su(2)$ is principally embedded in $su(3)$. For the coset central charge we find $c(m) = 2 \left(1 - \frac{12}{(m+2)(m+6)}\right)$. This indeed agrees with the series for the Fateev-Zamolodchikov spin-4/3 parafermionic algebra which is of type $\mathcal{W}(2, \frac{4}{3}, \frac{4}{3}, \dots)$ [115]. Let us look at the underlying (bosonic) \mathcal{W} -algebra. First we study the $m \rightarrow \infty$ limit $\mathcal{W}_c[A_2^{(1)}/A_1, 1]$. To count the number of $su(2)$ singlets $\widetilde{\Phi}_0^{\Lambda_0}(q)$ in the level-1 $\widehat{su}(3)$ vacuum module $\widetilde{L}_{\Lambda_0}$ we use the fact that under $\widehat{su}(2)$ we have a decomposition

$$\widetilde{L}_{\Lambda_0} = L_{4\Lambda_0} \oplus L_{4\Lambda_1}. \quad (7.65)$$

²⁸ For $N = 3$ one rather takes $su(2) \subset su(3)$ which has Dynkin index $j = 4$.

Hence

$$\begin{aligned}
\tilde{\Phi}_0^{\Lambda_0}(q) &= \Phi_0^{4\Lambda_0}(q) + \Phi_0^{4\Lambda_1}(q) \\
&= q^{\frac{1}{12}} \left(c_{4\Lambda_0}^{4\Lambda_0}(q) + c_{4\Lambda_0}^{4\Lambda_1}(q) - 2q^{\frac{1}{4}} c_{2\Lambda_0+2\Lambda_1}^{4\Lambda_0}(q) \right) \\
&= \frac{1}{F_2 F_3 F_4 F_5 F_6^2} (1 - 2q^9 - 7q^{10} + \mathcal{O}(q^{11})).
\end{aligned} \tag{7.66}$$

which indicates that $\mathcal{W}_c[A_2^{(1)}/A_1, 1]$ and thus $\mathcal{W}_c[A_2^{(1)} \oplus A_1^{(1)}/A_1^{(1)}, (1, m)]$, for large enough m , is of type $\mathcal{W}(2, 3, 4, 5, 6_2)$. In fact, analysing the branching functions of the coset pair $(A_2^{(1)} \oplus A_1^{(1)}, A_1^{(1)})$ itself, one finds that the coset is of type $\mathcal{W}(2, 3, 4, 5, 6_2)$ for $m \geq 3$. For $m = 1$, where the coset has central charge $c = \frac{6}{7}$, the same analysis suggests a \mathcal{W} -algebra of type $\mathcal{W}(2, 5)$. In fact $c = \frac{6}{7}$ is precisely one of the c -values for which there exists an exotic $\mathcal{W}(2, 5)$ -algebra [61] (see also Section 5.2.3). For $m = 2$ one finds type $\mathcal{W}(2, 3, 4, 5)$.

7.4.2. Dual coset pairs

Suppose we have two coset pairs $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{k}})$ and $(\widehat{\mathfrak{g}}', \widehat{\mathfrak{k}}')$. One can ask when their corresponding \mathcal{W} -algebras $\mathcal{W}_c[\widehat{\mathfrak{g}}/\widehat{\mathfrak{k}}, k]$ and $\mathcal{W}_c[\widehat{\mathfrak{g}}'/\widehat{\mathfrak{k}}', k']$ are isomorphic. This will evidently be the case when all the branching functions are in 1–1 correspondence. Such coset pairs were termed “dual” in [210].

A generic method to obtain dual coset pairs is by constructing so-called “ T -equivalent” coset pairs [71,5].²⁹ Two coset pairs $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{k}})$ and $(\widehat{\mathfrak{g}}', \widehat{\mathfrak{k}}')$ are called T -equivalent when there exists a Lie algebra \mathfrak{g}'' such that the embeddings $\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{k}}' \subset \widehat{\mathfrak{g}}''$ and $\widehat{\mathfrak{g}}' \oplus \widehat{\mathfrak{k}} \subset \widehat{\mathfrak{g}}''$ are conformal. This implies in particular that³⁰

$$T_1^{\mathfrak{g}''}(z) = T_{j(\mathfrak{g}'':\mathfrak{g})}^{\mathfrak{g}}(z) + T_{j(\mathfrak{g}'':\mathfrak{k}')}'(z) = T_{j(\mathfrak{g}'':\mathfrak{g}')}'(z) + T_{j(\mathfrak{g}'':\mathfrak{k})}^{\mathfrak{k}}(z). \tag{7.67}$$

Or, by swapping T 's around,

$$T_{j(\mathfrak{g}'':\mathfrak{g})}^{\mathfrak{g}/\mathfrak{k}}(z) = T_{j(\mathfrak{g}'':\mathfrak{g}')}'^{\mathfrak{g}'/\mathfrak{k}'}(z), \tag{7.68}$$

i.e. T -equivalent coset pairs have equivalent stress-energy tensors.

Examples of T -equivalent coset pairs were given in [71,5]. Apart from several exceptional cases there are four basic series:

²⁹ This, however, does not produce *all* dual coset pairs.

³⁰ Note that the level of $\widehat{\mathfrak{g}}''$ is necessarily equal to one. For clarity we have displayed the various levels by subscripts on $T^{\mathfrak{g}}(z)$. The Dynkin index of $\mathfrak{k} \subset \mathfrak{g}$ is denoted by $j(\mathfrak{g} : \mathfrak{k})$.

(I)

$$(su(2n)_m, sp(2n)_m) \cong (su(2m)_n, u(m)_n)$$

(II)

$$(su(N)_m \oplus su(N)_n, su(N)_{m+n}) \cong (su(m+n)_N, su(m)_N \oplus u(n)_N)$$

(III)

$$(so(N)_m \oplus so(N)_n, so(N)_{m+n}) \cong (so(m+n)_N, so(m)_N \oplus so(n)_N)$$

(IV)

$$(sp(2N)_m \oplus sp(2N)_n, sp(2N)_{m+n}) \cong (sp(2(m+n))_N, sp(2m)_N \oplus sp(2n)_N)$$

Here we have denoted the respective levels by subscripts. The “master Lie algebra” \mathfrak{g}'' in these four cases is given by (I) $so(4mn)_1$, (II) $su((m+n)N)_1$, (III) $so((m+n)N)_1$ and (IV) $so(4(m+n)N)_1$.

The prototype example of a T -equivalent coset pair is obtained by putting $N = m = 1$ in the series (IV). This yields two equivalent ways of producing the Virasoro unitary series (2.26) [169]. Another interesting and useful example occurs by putting $m = n = 1$ in the series (II). This example shows that the “ $su(2)$ parafermionic coset” $(su(2)_N, u(1))$ (also called \mathbb{Z}_N parafermions) [113,160] is T -equivalent to the \mathcal{W}_N diagonal coset $(su(N)_1 \oplus su(N)_1, su(N)_2)$. This observation, which was first made in [12], is very useful, for example, in constructing free field realizations for the $su(2)$ parafermions [177].

By iterative use of the four basic series (I)–(IV) one can generate a multitude of other interesting T -equivalent coset pairs. For instance, summing (II) for $m = 1$, we find

$$\bigoplus_{k=1}^n (su(N)_1 \oplus su(N)_k, su(N)_{k+1}) \cong \bigoplus_{k=1}^n (su(k+1)_N, su(k)_N \oplus u(1)). \quad (7.69)$$

Cancelling various terms on the right hand side of (7.69) yields a description of the $su(n+1)$ generalized parafermions [158] in terms of sums of \mathcal{W}_N models, generalizing the $su(2)$ case above

$$(su(n+1)_N, u(1)^n) \cong \bigoplus_{k=1}^n (su(N)_1 \oplus su(N)_k, su(N)_{k+1}).$$

This explains a curious relation between the central charges of the generalized parafermion and \mathcal{W}_N discrete series [108].

Cancelling also terms on the left hand side of (7.69) gives

$$(su(n+1)_N, u(1)^n) \cong \underbrace{(su(N)_1 \oplus \dots \oplus su(N)_1, su(N)_{m+1})}_{m+1},$$

which is useful for the construction of a generalized parafermion free scalar field realization [177,101].

As a consequence of the conformal embedding in $\widehat{\mathfrak{g}}''$ the branching functions of T -equivalent coset pairs satisfy a linear relation of the form

$$\sum_{\Lambda} n_{\Lambda\lambda'} b_{\lambda}^{\Lambda}(\mathfrak{g}/\mathfrak{k})(q) = \sum_{\Lambda'} m_{\Lambda'\lambda} b_{\lambda'}^{\Lambda'}(\mathfrak{g}'/\mathfrak{k}')(q), \quad (7.70)$$

for some integers $n_{\Lambda\lambda'}$ and $m_{\Lambda'\lambda}$. To establish that these T -equivalent coset pairs give rise to a set of dual coset pairs, *i.e.* coset pairs having the same set of branching functions, requires an explicit determination of the integers $n_{\Lambda\lambda'}$ and $m_{\Lambda'\lambda}$. Duality has been established in the cases (I) $m = 1$ [209,5], (II) $m = n = 1$ [5], (IV) $N = 1$ [218,71,5].

8. Further developments

8.1. \mathcal{W} -gravity

In this section we will briefly review some developments in (classical and quantum) \mathcal{W} -gravity. \mathcal{W} -gravity is a higher-spin extension of $d = 2$ gravity whose structure is based on an underlying \mathcal{W} -algebra. This \mathcal{W} -algebra takes over the role played by the Virasoro algebra in pure $d = 2$ gravity. The main applications of \mathcal{W} -gravity, which are in the area of string theory, will be discussed in the next section. Review papers on \mathcal{W} -gravity are for example [33,196,309].

It is well-known that classical theories of gravity or supergravity (in general dimensions) can be constructed in a systematic way by starting from an algebra of space-time (super)symmetries. This is done by first constructing a gauge theory for the symmetry algebra and then imposing a number of Yang-Mills curvature constraints. Because of these constraints general coordinate transformations become a local symmetry, and in this way a theory of (super)gravity arises [220]. For $d = 2$ \mathcal{W} -gravity this procedure was successfully applied in [305], where a covariant Lagrange formulation of classical \mathcal{W}_3 gravity coupled to scalar fields was presented.

The gauge algebra used in [305] for the construction of covariant \mathcal{W}_3 gravity is a centerless classical limit of the quantum \mathcal{W}_3 algebra. After solving the Yang-Mills curvature constraints in the corresponding gauge theory, one finds a gauge multiplet which contains four zweibein fields e_μ^+ , e_μ^- and four \mathcal{W}_3 zweibein fields b_μ^{++} , b_μ^{--} , where $\mu = \pm$. There are eight local gauge symmetries: general coordinate, Weyl and local Lorentz symmetries, together with their \mathcal{W}_3 analogues. The coupling of these \mathcal{W}_3 gravity fields to scalar matter fields was worked out in [305], where a gauge invariant kinetic action for N scalar fields ϕ^i , $i = 1, 2, \dots, N$ was presented. We shall not display this action here, but instead discuss the results in the chiral gauge and in the light-cone gauge, which can both be obtained by fixing some of the local symmetries.

In the chiral gauge the coupling of scalar matter fields to \mathcal{W}_3 gravity, which was first obtained in [192], is easily explained. We start from the observation [192] that a free action for N scalar fields admits a rigid \mathcal{W}_3 symmetry. The transformations of the scalar fields read

$$\delta_\epsilon \phi^i = \epsilon_+ \partial_- \phi^i, \quad \delta_\lambda \phi^i = \lambda_{++} d^{ijk} \partial_- \phi^j \partial_- \phi^k, \quad (8.1)$$

where d^{ijk} is a symmetric 3-index tensor. [We write ∂_- and ∂_+ for ∂_z and $\partial_{\bar{z}}$, respectively.] One can promote these rigid symmetries to local gauge invariances by introducing gauge fields h_{++} and b_{+++} in the standard way. It turns out [192] that the scalar field action with only the minimal coupling to these gauge fields,

$$S_{\text{ch}} = \frac{1}{\pi} \int d^2z \left[-\frac{1}{2} \partial_+ \phi^i \partial_- \phi^i + \frac{1}{2} h_{++} \partial_+ \phi^i \partial_+ \phi^i + \frac{1}{3} b_{+++} d^{ijk} \partial_+ \phi^i \partial_+ \phi^j \partial_+ \phi^k \right], \quad (8.2)$$

is gauge-invariant, provided we choose the transformation rules of h_{++} and b_{+++} appropriately and we have the identity

$$d^{k(ij} d^{l)mk} = \delta^{(ij} \delta^{l)m}. \quad (8.3)$$

The two local symmetries, with parameters $\epsilon_+(z, \bar{z})$ and $\lambda_{++}(z, \bar{z})$, can be viewed as particular linear combinations of all eight local symmetries present in the covariant formulation.

In the light-cone gauge, with both chiralities present, the coupling of scalar matter fields to \mathcal{W}_3 gravity is more involved. The \mathcal{W}_3 gauge fields are $h_{\pm\pm}$ and $b_{\pm\pm\pm}$, corresponding to local symmetries with parameters $\epsilon_{\pm}(z, \bar{z})$ and $\lambda_{\pm\pm}(z, \bar{z})$. In [304] it was found that the light-cone gauge action is non-polynomial in the spin-3 gauge fields. This difficulty can be circumvented by introducing auxiliary fields $F_+{}^i$ and $F_-{}^i$, which will play the role of so-called *nested covariant derivatives*. With these variables, the gauge-invariant action takes the following form [304]

$$S_{\text{lc}} = \frac{-1}{\pi} \int d^2z e \left[-\frac{1}{2} \nabla_+ \phi^i \nabla_- \phi^i - F_+{}^i F_-{}^i + F_+{}^i \left(\nabla_- \phi^i - \frac{1}{3} b_{---} d^{ijk} F_+{}^j F_+{}^k \right) + F_-{}^i \left(\nabla_+ \phi^i - \frac{1}{3} b_{---} d^{ijk} F_-{}^j F_-{}^k \right) \right], \quad (8.4)$$

where $e = (1 - h_{++} h_{--})^{-1}$ and $\nabla_{\pm} = \partial_{\pm} - h_{\pm\pm} \partial_{\mp}$. The field equation of $F_-{}^i$ is algebraic and leads to

$$F_+{}^i = \nabla_+ \phi^i - b_{+++} d^{ijk} F_-{}^j F_-{}^k \quad (8.5)$$

(with a similar result for $F_-{}^i$). When solving $F_{\pm}{}^i$ by iteration one obtains a \mathcal{W}_3 generalization of a covariant derivative, which is infinitely nonlinear and is appropriately called a nested covariant derivative.

In a similar way, light-cone gauge and covariant actions for w_{∞} gravity coupled to scalar fields were constructed in [38] and in [306], respectively. A geometrical interpretation of the full non-linear structure of w_{∞} gravity was presented in [194]. We refer to [33,193,252] for further results on classical \mathcal{W} -gravity.

At the classical level, \mathcal{W} -gravity does not have degrees of freedom, since all the field components are compensated for by local gauge symmetries (up to possible moduli). However, upon quantization one or more of these symmetries become anomalous and therefore quantum \mathcal{W} -gravity does have degrees of freedom. If the central charge c of the matter system coupled to \mathcal{W} -gravity is tuned to a specific value, the quantum anomalies cancel and the \mathcal{W} -gravity degrees of freedom decouple. This is similar to what happens in $d = 2$ induced gauge theories and $d = 2$ ordinary gravity [279,309].

The dynamics of the quantum degrees of freedom of \mathcal{W} -gravity are governed by an *effective* action. This action can be studied for various gauge choices, for example in the chiral gauge or in the conformal gauge. In the chiral gauge the effective action arises in the following way [251,307,195,308,178].

One first obtains an *induced* action by performing a functional integral over the matter fields, of total central charge c , to which the \mathcal{W} -gravity fields are coupled. The non-trivial contributions to the induced action arise from loop-diagrams with \mathcal{W} -gravity fields on external lines and matter fields in the loops. The resulting induced action, which can be viewed as an integrated anomaly, is non-local. The *effective* action is then obtained by renormalizing this induced action, taking into account loop-diagrams for the anomalously propagating \mathcal{W} -gravity fields. The entire computation of the effective action can be done perturbatively, with $1/c$ as the expansion parameter.

In [86,87] the covariant induced action for \mathcal{W}_n gravity (in the limit $c \rightarrow \pm\infty$) was presented. It is given in terms of two (left and right) chiral sectors, which contain fields $b_{\pm}^{(i)}$, $i = 2, \dots, n$, $(n - 1)$ scalar fields ϕ^i , $i = 2, \dots, n$, and a number of auxiliary fields which play the role of nested covariant derivatives as in (8.4), (8.5). When specializing to the conformal gauge, one finds [86] that this induced action takes the form of a Toda action for the scalar fields ϕ^i . This implies that in the conformal gauge the \mathcal{W} -currents of \mathcal{W} -gravity take the familiar free-field form (compare with (6.45), (6.53)). Because of this, explicit computations in quantum \mathcal{W} -gravity, such as for example the analysis of the BRST cohomology of physical states, are most easily done in the conformal gauge [84].

In the above we mentioned that the effective action for \mathcal{W} -gravity can be computed perturbatively in quantum field theory, with $1/c$ as the expansion parameter. However, it is possible to exploit once more the relation between \mathcal{W} -algebras and affine Lie algebras to obtain results exact to all orders for some of the quantities in \mathcal{W} -gravity! The idea here is that \mathcal{W} -gravity can be obtained by constraining WZW field theory or, equivalently, by

reducing $d = 3$ Chern-Simons gauge theory. We will now briefly explain how this works for \mathcal{W}_3 gravity.

Consider \mathcal{W}_3 gravity coupled to a \mathcal{W}_3 CFT of central charge c . The matter sector can be described in the QDS reduction scheme (see Chapter 6), where the starting point is the affine Lie algebra $\widehat{sl}(3)$ at level k , with (see (6.13))

$$c = c_k = 2 - 24 \left(\sqrt{\frac{1}{k+3}} - \sqrt{k+3} \right)^2 = 50 - 24 \left(\frac{1}{k+3} + k + 3 \right). \quad (8.6)$$

To couple this system to \mathcal{W}_3 gravity, we must make sure that the central charge of the \mathcal{W}_3 gravity sector compensates the matter central charge c_k *plus* the contribution of the \mathcal{W}_3 ghosts, which equals $c_{gh} = -100$ (see Section 8.2). Observe that $c_k = 100 - c_{-(k+6)}$, which shows that we obtain the correct contribution to the central charge if we apply the QDS reduction to an $sl(3)$ WZW model of level $\kappa = -(k+6)$. With (8.6) this gives

$$\kappa = -\frac{1}{48} \left(50 - c + \sqrt{(c-98)(c-2)} \right) - 3, \quad (8.7)$$

where the sign in front of the square root has been chosen in accordance with the classical limit $c \rightarrow -\infty$, in which $\kappa \sim c/24$.

The picture that arises at this point is the following: we can represent the degrees of freedom of the *effective* quantum \mathcal{W}_3 gravity, induced from a matter system of central charge c and after renormalization, by a QDS reduced $sl(3)$ WZW model of level κ as in (8.7). This observation explains the presence of a ‘hidden’ affine $sl(3)$ symmetry in \mathcal{W}_3 gravity, which generalizes the affine $sl(2)$ symmetry in ordinary gravity first observed by Polyakov [278]. For ordinary gravity the connection with a constrained $sl(2)$ WZW model was first understood in [233]; the generalization to $sl(n)$ was worked out in [42].

From the above picture one expects that the effective action for \mathcal{W}_n -gravity can be obtained in explicit form by reducing the $sl(n)$ WZW action. For \mathcal{W}_3 gravity in the chiral gauge this result has been checked by explicit perturbative computations. It was found [251,48,83,274] that the induced action for chiral \mathcal{W}_3 gravity, in the limit $c \rightarrow \pm\infty$, is governed by local Ward identities which can be obtained by reducing the Ward identities for the $sl(3)$ WZW model. For finite c , the induced action contains additional terms that correspond to non-local additional terms in the Ward identities [307]. However, the results of [308] suggest that these extra terms get cancelled if one renormalizes the induced action. In [308] the full effective action was computed through the first non-leading order in the

perturbative $1/c$ expansion, and it was found that the result precisely takes the form of a reduced WZW action with renormalized coefficients. The overall coefficient of the effective action can be identified with the level κ of the affine $sl(3)$ algebra. The perturbative result for κ thus obtained agrees with the relation (8.7) through the first non-leading order in $1/c$.

In [274,309] the effective action for chiral \mathcal{W}_3 gravity was given in a closed form. In terms of Polyakov type variables f and g , this action is *local*, and it generalizes the effective action for ordinary gravity in terms of Polyakov's variable f [278].

8.2. \mathcal{W} -symmetry in string theory

We would now like to come back to some remarks we made in Chapter 1, namely the applications of \mathcal{W} -symmetry in string theory. Many of the important issues in this field are still unsettled and the discussion below is intended to merely give a flavor of the developments that are taking place.

In the standard formulation of string theory, the fields representing the coordinates of a (first-quantized) string in target space-time, define a 'matter' CFT. The reparametrization invariance on the string world-sheet implies that this CFT should be coupled to two-dimensional (world-sheet) gravity. If the total central charge of the 'matter' CFT is equal to 25, the $d = 2$ gravity sector decouples. In that case the gravity sector adds one free scalar field to the space-time coordinates, and it leads to a number of constraints (the Virasoro constraints) on physical states in the string theory. If the matter central charge differs from 25 the extra scalar field becomes interacting and the situation is more intricate.

In recent years, considerable progress has been made in understanding the coupling of $c \leq 1$ (minimal) CFT's to $d = 2$ gravity. It has turned out that these theories can also be studied from the point of view of discretized world-sheets (matrix models) or from the point of view of topological field theories. For $c > 1$, which is the regime where the bosonic string develops tachyonic states, the coupling to $d = 2$ gravity has run into 'strong-coupling problems' [233] and has not yet been understood properly.

In the study of $c < 1$ CFT's coupled to gravity, interesting connections with \mathcal{W} -symmetry have come up. Namely, it was found, both in the matrix model formulation and in the topological approach, that the partition function of the theory (as a function of a set of coupling constants) can be characterized by so-called \mathcal{W} -constraints [96,150,174]. The appearance of these constraints is closely related to the fact that these models can

be analyzed in terms of generalized KdV hierarchies [99]. The paper [151] shows that the \mathcal{W}_n constraints can be viewed as reductions of more general $\mathcal{W}_{1+\infty}$ constraints, which arise naturally when one views the generalized KdV hierarchies as reductions of the KP hierarchy.

The study of $c = 1$ strings, both in terms of matrix models and as a continuum theory, has revealed an interesting symmetry structure. In the case of a flat, uncompactified background the (chiral) symmetries in the continuum theory have been identified [230,336] with the area preserving polynomial vector-fields, generating (the wedge of) a w_∞ algebra as in (5.3).

In the cases just mentioned \mathcal{W} -symmetries arise as ‘bonus’ symmetries in specific string theories that were constructed without any reference to \mathcal{W} -symmetry. It is certainly interesting to see if one can construct string theories with manifest \mathcal{W} -symmetries built in. A first step in this direction is to study \mathcal{W} -extensions of $d = 2$ gravity and their couplings to CFT. In Section 8.1 we briefly reviewed some results in this area. We will now further explore the possibilities to construct \mathcal{W} -extensions of string theories.

A first remark concerns the \mathcal{W} -constraints in matrix models and topological field theories mentioned above. It has been proposed [7,84] that the coupled systems of CFT plus gravity, for which these constraints occur, are actually closely related to theories of *pure* \mathcal{W} -gravity. Physical states in the spectrum of pure \mathcal{W} -gravity can sometimes be viewed as CFT matter states ‘dressed’ by the fields of pure $d = 2$ gravity [84]. However, full equivalence of both theories cannot be claimed, and the connection remains rather mysterious.

Going one step further, one can consider \mathcal{W} -invariant CFT’s coupled to the corresponding \mathcal{W} -gravity. One interesting observation is that in this context the critical central charge $c = 1$ for ordinary gravity gets shifted to a higher value. For example, for the \mathcal{W} -algebras related to the *ADE* simply-laced Lie algebras the threshold value for the central charge would be the rank ℓ of the Lie algebra. [For \mathcal{W}_3 , where $\ell = 2$, this can be read off from equation (8.7); other cases can be treated similarly.] This makes clear that certain CFT’s with $c > 1$, whose coupling to ordinary gravity is problematic, can consistently be coupled to \mathcal{W} -gravity. We expect that such can be done for *any* RCFT, where the \mathcal{W} -algebra for the \mathcal{W} -gravity theory should be closely related to the chiral algebra of the RCFT.

Systems of \mathcal{W} -invariant CFT coupled to \mathcal{W} -gravity can tentatively be interpreted as

(critical or non-critical) \mathcal{W} -strings. For the case of $c < \ell$ minimal models of one of the *ADE* \mathcal{W} -algebras coupled to the corresponding theory of \mathcal{W} -gravity, one expects behavior which is qualitatively similar to that of the standard $c < 1$ strings. In particular, one expects connections with generalized matrix models and topological field theories, which still remain to be worked out.

The direct generalization of lower critical ($c_1 = 1$) and upper critical ($c_2 = 26$) bosonic strings to \mathcal{W} -strings is problematic. On the level of the algebra the numerology is clear: for example, for the *ADE* \mathcal{W} -algebras the lower critical dimension is $c_1 = \ell$, and the upper critical dimension has been found to be $c_2 = 2\ell(2h^2 + 2h + 1)$ [11], where h is the Coxeter number of the Lie algebra. The latter value is the one for which a nilpotent BRST charge is expected to exist. For a general (generic) \mathcal{W} -algebra, the critical central charge is given by

$$c = \sum_s 2(-1)^{2s} (6s^2 - 6s + 1) , \quad (8.8)$$

where the summation runs over the spins s of the independent generators of the \mathcal{W} -algebra. However, the existence of a nilpotent BRST charge is not always guaranteed (due to complications with non-linearity, see [303]) and should be checked in each individual case. For the \mathcal{W}_3 algebra, where $c_2 = 100$, a nilpotent quantum BRST charge has been constructed in [319,303]. [For infinite \mathcal{W} -algebras the sum (8.8) needs to be regularized. It was argued in [340] that for the \mathcal{W}_∞ algebra a nilpotent BRST charge exists for $c = -2$.]

The facts that the notion of critical central charges $c_1 = 1$ and $c_2 = 26$ can be generalized to (at least) the *ADE* \mathcal{W} -algebras, and that nilpotent BRST charges can presumably be constructed, do not by themselves imply the existence of interesting new string theories. In addition, a string theory requires that (part of) the matter conformal field theory can be viewed as a set of coordinates on a target space-time. In practice one should therefore consider realizations of (critical or non-critical) \mathcal{W} -algebras in terms of scalar matter fields. For the \mathcal{W}_3 algebra realizations in terms of an arbitrary number of scalar fields (and with adjustable central charge) were discussed in [293]. The anomaly-free coupling of such matter systems with central charge $c = 100$ to \mathcal{W}_3 gravity was discussed in [286].

In general, scalar field realizations of \mathcal{W} -algebras involve background charges, which lead to mass-shifts in the spectrum of physical string states. Because of that, the original expectation that the spectrum of a \mathcal{W} -string might contain states with space-time spin greater than two [53] is probably not justified. Also, the fact that the equations determining

the BRST cohomology of physical states involve higher (≥ 2) order polynomials in general, leads to a ‘branching’ of the spectrum of physical states. As a consequence, in the lower critical dimension $d = \ell$, \mathcal{W} -strings for the *ADE* \mathcal{W} -algebras are *not* free from tachyons [84]. In fact, it was argued in [84] that the critical numbers d_1 and d_2 of scalar fields in \mathcal{W}_n gravity (as opposed to the critical central charges) are $d_1 = 6/n(n+1)$ and $d_2 = 24 + 6/n(n+1)$. We would like to stress, however, that the validity of these results depends crucially on a specific (but debatable) Ansatz for the matter couplings in the conformal gauge.

The spectrum of critical \mathcal{W}_n strings has been further analyzed in [239]. In [238] some of these results were generalized to \mathcal{W} -strings based on more general \mathcal{W} -algebras.

String theory is more geometrical than CFT, which seems to be one of the reasons why the application of finitely generated \mathcal{W} -algebras to string theory is rather intricate. There have been a number of proposals for a more geometrical understanding of \mathcal{W} -symmetry, but their possible applications to string theory have not been clarified.

Some related ideas concerning the geometrical structure of \mathcal{W} -symmetries have been proposed in the papers [316,315,164], see also [46,162,48]. The paper [164] associates \mathcal{W} -symmetries with the extrinsic geometry of the embedding of two-dimensional manifolds with chiral parametrization into higher dimensional Kähler manifolds. The characteristic equations for such embeddings can be connected to a Lax pair for certain Toda equations, and these then form the link to \mathcal{W} -symmetries.

The constructions of \mathcal{W} -gravity based on Drinfeld-Sokolov reduction and on the connection with $d = 3$ Chern-Simons theory (see Section 8.1) suggest an alternative way to understand the geometry of \mathcal{W} -symmetries. These ideas have been essential for the explicit construction of the covariant induced action of \mathcal{W}_n gravity [87].

In conclusion, we would like to stress that the status of \mathcal{W} -symmetry in the context of CFT is much better understood than the role these symmetries can play in string theory. The precise interpretation, in the context of string theory, of results obtained for \mathcal{W} -gravity and \mathcal{W} -geometry is not always clear. However, the fact that already ‘toy models’ of $c < 1$ and $c = 1$ strings exhibit \mathcal{W} -symmetries certainly suggests that extended symmetries will eventually also play a role in more realistic theories of first and second quantized strings.

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Note added. An interesting problem that has not been solved is to generalize the explicit expression for the coset spin-3 generator, as discussed in section 7.3.2, to the other coset generators of spin greater than 3. In the preprint version of this paper we gave some expressions for these which are however incorrect. The problem can be traced back to a higher-spin generalization of (7.26) that was first proposed in [320] (equation (2.12)). Inconsistencies in the coset expressions derived from this equation show that it cannot be correct (see also the erratum to [326]). It is now understood that the equation fails to hold because it assumes a tensor identity for higher order d -symbols which is simply not valid. We would like to thank G. Watts for pointing out this problem and A. Sudbery for correspondence on the tensor identities.

9. Appendices

A. Lie algebra conventions

Throughout the report we use the following conventions (see *e.g.* [214] for more details):

\mathfrak{g}	a finite-dimensional complex semi-simple Lie (super)algebra
\mathfrak{h}	the Cartan subalgebra (CSA) of \mathfrak{g}
\mathfrak{h}^*	the dual CSA; we will identify \mathfrak{h} with \mathfrak{h}^*
$\mathfrak{g} \cong \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$	a Cartan (triangular) decomposition of \mathfrak{g}
(\cdot, \cdot)	bilinear form on \mathfrak{h}^* (sometimes also denoted by a simple dot \cdot), normalized such that $(\alpha, \alpha) = 2$ for a long root of \mathfrak{g}
ℓ	the rank of \mathfrak{g}
$\dim \mathfrak{g}$	the dimension of \mathfrak{g}
Δ_+	set of positive roots α of \mathfrak{g}
α_i	a simple root of \mathfrak{g}
Λ_i	a fundamental weight of \mathfrak{g}
ρ	the Weyl vector of \mathfrak{g}
h	the Coxeter number of \mathfrak{g}
Q	the long root lattice of \mathfrak{g}
P_+	the set of integral dominant weights $\lambda \in \mathfrak{h}^*$
W	the Weyl group of \mathfrak{g}
$l(w)$	the length of the Weyl group element w
$\epsilon(w)$	the determinant (<i>i.e.</i> ± 1) of the Weyl group element w
r_α (r_i)	reflection in the root $\alpha \in \Delta_+$ (α_i)
$\mathcal{U}(\mathfrak{g})$	the universal enveloping algebra of \mathfrak{g}
$\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$	the center of $\mathcal{U}(\mathfrak{g})$
h^i	a basis of the CSA \mathfrak{h}
e^α	Lie algebra element corresponding to root α
a_{ij}	the Cartan matrix of \mathfrak{g} , <i>i.e.</i> $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$
$e_i, i = 1, \dots, \ell$	the exponents of \mathfrak{g}

The dual Lie algebra \mathfrak{g}^* is the algebra obtained by inverting the arrows in the Dynkin diagram corresponding to \mathfrak{g} . Its corresponding characteristics are denoted by a subscript

\vee , i.e. $\rho^\vee, \mathfrak{h}^\vee, P_+^\vee$ etc. In particular the roots α^\vee of \mathfrak{g}^* are related to those of \mathfrak{g} by $\alpha^\vee = 2\alpha/(\alpha, \alpha)$.

The untwisted affine Lie algebra $\mathfrak{g} \otimes \mathcal{C}[t, t^{-1}] \oplus \mathcal{C}c$ [214] corresponding to \mathfrak{g} will be denoted by either $\widehat{\mathfrak{g}}$ or $\mathfrak{g}^{(1)}$. Characteristics of the affine Lie algebra $\widehat{\mathfrak{g}}$ and the underlying finite-dimensional Lie algebra \mathfrak{g} will be distinguished by putting hats on the former. In addition we will identify affine weights $\widehat{\Lambda}$ with their finite-dimensional projection Λ supplied by the level k . We use the notation $P_+^{(k)}$ for the set of integral dominant weights of level k . The twisted length $\tilde{l}(w)$ of an affine Weyl group element $w \in \widehat{W}$ is defined in *e.g.* [65].

We have collected some of the characteristics of finite-dimensional simple Lie algebras \mathfrak{g} in Table 1.

\mathfrak{g}	$\dim \mathfrak{g}$	h	\mathfrak{h}^\vee	$\{e_i\}$
A_n	$n(n+2)$	$n+1$	$n+1$	$1, 2, \dots, n$
B_n	$n(2n+1)$	$2n$	$2n-1$	$1, 3, 5, \dots, 2n-1$
C_n	$n(2n+1)$	$2n$	$n+1$	$1, 3, 5, \dots, 2n-1$
D_n	$n(2n-1)$	$2(n-1)$	$2(n-1)$	$1, 3, 5, \dots, 2n-3, n-1$
E_6	78	12	12	$1, 4, 5, 7, 8, 11$
E_7	133	18	18	$1, 5, 7, 9, 11, 13, 17$
E_8	248	30	30	$1, 7, 11, 13, 17, 19, 23, 29$
F_4	52	12	9	$1, 5, 7, 11$
G_2	14	6	4	$1, 5$

Table 1

Finally, we collect here some useful formulae [214]

$$\begin{aligned} \rho &= \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha, & (\rho, \alpha_i^\vee) &= 1, \\ \rho^\vee &= \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha^\vee, & (\rho^\vee, \alpha_i) &= 1, \end{aligned} \tag{9.1}$$

$$\dim \mathfrak{g} = \ell(1 + h). \tag{9.2}$$

The denominator formula:

$$\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha}) = \sum_{w \in W} \epsilon(w) e^{w\rho - \rho}. \tag{9.3}$$

The Freudenthal-de Vries strange formula:

$$\frac{|\rho|^2}{2h^\vee} = \frac{\dim \mathfrak{g}}{24}. \quad (9.4)$$

And some relations for the exponents:

$$\begin{aligned} \sum e_i &= \frac{1}{2}\ell h, \\ \sum e_i(e_i + 1) &= 4(\rho, \rho^\vee), \\ &= \frac{1}{3}\ell h(h + 1) \quad \text{for } \mathfrak{g} \text{ simply laced.} \end{aligned} \quad (9.5)$$

B. \mathcal{W} -algebra nomenclature

In this appendix we propose some systematics for naming \mathcal{W} -algebras. Obviously, this proposal comes somewhat *après la date*, but we think that it is worthwhile to try and standardize these matters. We base our notations for \mathcal{W} -algebras on the different approaches to their construction, which we discuss in the Chapters 5, 6 and 7, respectively.

1. We introduce the notion of a \mathcal{W} -algebra of *type* $\mathcal{W}(2, s_2, s_3, \dots, s_n)$. This refers to an algebra that is generated (in the sense of our definition in Section 3.1) by the Virasoro generator $T(z)$ (which is quasi-primary of spin 2) and additional primary currents of spins s_2, s_3, \dots, s_n . For a N -extended \mathcal{W} -superalgebra (which contains the $O(N)$ -extended superconformal algebra, $N = 1, 2, 3$ or 4), we write $\mathcal{SW}^{(N)}(2 - \frac{N}{2}, s_2, \dots, s_n)$. The first entry refers to the super-Virasoro generator (which is quasi-primary of spin $2 - \frac{N}{2}$) and the additional entries s_2, s_3, \dots, s_n refer to additional generating currents, which are all superfields in N -extended chiral superspace [300].

Clearly, the type of a \mathcal{W} -algebra does not always completely fix the algebra. There can be free parameters (the central charge or still others) and there is a possibility of entirely distinct algebras with the same set of spins of the generating currents.

2. In the most general version of the Drinfeld-Sokolov scheme (see Section 6.3), a \mathcal{W} -algebra is completely determined by a triple $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi)$, consisting of an affine Lie algebra $\widehat{\mathfrak{g}}$, an affine subalgebra $\widehat{\mathfrak{g}}' \subset \widehat{\mathfrak{g}}$ and a 1-dimensional representation χ of $\widehat{\mathfrak{g}}'$. We denote the corresponding \mathcal{W} -algebra by $\mathcal{W}_{DS}[\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi]$. In many cases one makes a special choice for the defining triple (namely $\widehat{\mathfrak{g}}' = \widehat{\mathfrak{n}}_+$, $\chi = \chi_{DS}$) which is completely determined by the embedding of an $sl(2)$ subalgebra in \mathfrak{g} . The \mathcal{W} -algebra corresponding to that situation

will be called $\mathcal{W}_{DS}[\widehat{\mathfrak{g}}, k, \delta]$, where k is the level of $\widehat{\mathfrak{g}}$ and the vector δ specifies how the $sl(2)$ subalgebra is embedded in \mathfrak{g} .

3. In the coset construction, discussed in Chapter 7, a \mathcal{W} -algebra is specified by a coset pair $\widehat{\mathfrak{g}}' \subset \widehat{\mathfrak{g}}$, where $\widehat{\mathfrak{g}}$ is an affine Lie algebra of level k and $\widehat{\mathfrak{g}}'$ is an affine sub-algebra, or by a pair $\mathfrak{g} \subset \widehat{\mathfrak{g}}$, where \mathfrak{g} is the finite dimensional horizontal subalgebra of $\widehat{\mathfrak{g}}$. (In the latter case the coset construction reduces to what we called the Casimir construction or extended Sugawara construction.) The \mathcal{W} -algebras for such coset pairs will be denoted by $\mathcal{W}_c[\widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}', k, *]$, where the $*$ specifies the embedding $\widehat{\mathfrak{g}}' \subset \widehat{\mathfrak{g}}$, and by $\mathcal{W}_c[\widehat{\mathfrak{g}}/\mathfrak{g}, k]$, respectively.

In many cases these notations can be simplified. The subscripts DS and c can be dropped if the context is clear. The embedding data δ and $*$ can be defaulted for ‘obvious’ choices, such as the principal $sl(2)$ embedding for δ , or, for $*$, the diagonal embedding $\widehat{\mathfrak{g}} \subset \widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}$.

Furthermore, there are ‘nicknames’ for the most familiar algebras. In particular, there are the \mathcal{W}_n algebras, of type $\mathcal{W}(2, 3, \dots, n)$, which can be realized as $\mathcal{W}_{DS}[A_{n-1}^{(1)}, k]$, as $\mathcal{W}_c[A_{n-1}^{(1)}/A_{n-1}, 1]$ for $c = n$, or as $\mathcal{W}_c[A_{n-1}^{(1)} \oplus A_{n-1}^{(1)}/A_{n-1}^{(1)}, (1, k)]$ for the central charges in the minimal series. We used the name super- \mathcal{W}_3 algebra for the algebra of type $\mathcal{SW}^{(1)}(3/2, 5/2)$ discussed in Section 3.3. Similarly, there are the $N = 2$ super- \mathcal{W}_n algebras, of type $\mathcal{SW}^{(2)}(1, 2, \dots, n)$, which can be realized as $\mathcal{W}_{DS}[A(n, n-1)^{(1)}, k]$ (see Section 6.3.3).

Algebras that are obtained by applying the DS scheme to various embeddings of $sl(2)$ in A_{n-1} are sometimes called $\mathcal{W}_n^{(l)}$ algebras, where the superscript indicates the embedding that is used (with $l = 1$ denoting the principal embedding).

In the literature the notation \mathcal{WX}_ℓ algebra is often used for a \mathcal{W} -algebra that one can associate with the Lie algebra X_ℓ . This notation works fine for the simply-laced Lie algebras $X_\ell = A_\ell, D_\ell$ or E_ℓ but it is confusing in other cases and we tried as much as possible to avoid it. We can explain our concern with the example $X_\ell = B_\ell$. DS reduction of the affine algebra $B_\ell^{(1)}$ leads to the \mathcal{W} -algebra $\mathcal{W}_{DS}[B_\ell^{(1)}, k]$, which is of type $\mathcal{W}(2, 4, 6, \dots, 2\ell)$. On the other hand, the coset \mathcal{W} -algebras $\mathcal{W}_c[B_\ell^{(1)}/B_\ell, 1]$ and $\mathcal{W}_c[B_\ell^{(1)} \oplus B_\ell^{(1)}/B_\ell^{(1)}, (1, k)]$ are all of type $\mathcal{W}(2, 4, 6, \dots, 2\ell, \ell + 1/2)$, which is the spin content that one would get by applying DS reduction to the superalgebra $B(0, \ell)^{(1)}$ rather than to $B_\ell^{(1)}$ itself.

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