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Drinfeld–Sokolov Reduction for Difference Operators and Deformations of $\mathcal{W}$-Algebras I. The Case of Virasoro Algebra

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Abstract: We propose a q-difference version of the Drinfeld-Sokolov reduction scheme, which gives us q-deformations of the classical $\mathcal{W}$-algebras by reduction from Poisson-Lie loop groups. We consider in detail the case of SL_2. The nontrivial consistency conditions fix the choice of the classical r-matrix defining the Poisson-Lie structure on the loop group LSL_2, and this leads to a new elliptic classical r-matrix. The reduced Poisson algebra coincides with the deformation of the classical Virasoro algebra previously defined in [19]. We also consider a discrete analogue of this Poisson algebra. In the second part [31] the construction is generalized to the case of an arbitrary semisimple Lie algebra.

1. Introduction

It is well-known that the space of ordinary differential operators of the form $\partial^n + u_1\partial^{n-2} + \ldots + u_{n-1}$ has a remarkable Poisson structure, often called the (second) Adler-Gelfand-Dickey bracket [1, 12]. Drinfeld–Sokolov reduction [11] gives a natural realization of this Poisson structure via the hamiltonian reduction of the dual space to the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_n$. Drinfeld and Sokolov [11] have applied an analogous reduction procedure to the dual space of the affinization $\widehat{\mathfrak{g}}$ of an arbitrary semisimple Lie algebra $\mathfrak{g}$. The Poisson algebra $\mathcal{W}(\mathfrak{g})$ of functionals on the corresponding reduced space is called the classical $\mathcal{W}$-algebra. Thus, one can associate a classical $\mathcal{W}$-algebra to an arbitrary semisimple Lie algebra $\mathfrak{g}$. In particular, the classical $\mathcal{W}$-algebra associated to $\mathfrak{sl}_2$ is nothing but the classical Virasoro algebra, i.e., the Poisson algebra of functionals on the dual space to the Virasoro algebra (see, e.g., [19]).

It is interesting that $\mathcal{W}(\mathfrak{g})$ admits another description as the center of the universal enveloping algebra of an affine algebra. More precisely, let $Z(\widehat{\mathfrak{g}})_{-h^\vee}$ be the center of a

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completion of the universal enveloping algebra $U(\widehat{\mathfrak{g}})_{-h^\vee}$ at the critical level $k = -h^\vee$ (minus the dual Coxeter number). This center has a canonical Poisson structure. It was conjectured by V. Drinfeld and proved by B. Feigin and E. Frenkel [14, 18] that as the Poisson algebra $Z(\widehat{\mathfrak{g}})_{-h^\vee}$ is isomorphic to the classical $\mathcal{W}$-algebra $\mathcal{W}({}^L\mathfrak{g})$ associated with the Langlands dual Lie algebra ${}^L\mathfrak{g}$ of $\mathfrak{g}$.

In [19] two of the authors used this second realization of $\mathcal{W}$-algebras to obtain their q-deformations. For instance, the q-deformation $\mathcal{W}_q(\mathfrak{sl}_n)$ of $\mathcal{W}(\mathfrak{sl}_n)$ was defined as the center $Z_q(\widehat{\mathfrak{sl}}_n)$ of a completion of the quantized universal enveloping algebra $U_q(\widehat{\mathfrak{sl}}_n)_{-h^\vee}$. The Poisson structure on $Z_q(\widehat{\mathfrak{sl}}_n)$ was explicitly described in [19] using results of [26]. It was shown that the underlying Poisson manifold of $Z_q(\widehat{\mathfrak{sl}}_n) = \mathcal{W}_q(\mathfrak{sl}_n)$ is the space of q-difference operators of the form $D_q^n + t_1 D_q^{n-1} + \ldots + t_{n-1} D_q + 1$. Furthermore, in [19] a q-deformation of the Miura transformation, i.e., a homomorphism from $\mathcal{W}_q(\mathfrak{sl}_n)$ to a Heisenberg-Poisson algebra, was defined. The construction [19] of $\mathcal{W}_q(\mathfrak{sl}_n)$ was followed by further developments: it was quantized [32, 15, 4] and the quantum algebra was used in the study of lattice models [25, 3]; the Yangian analogue of $\mathcal{W}_q(\mathfrak{sl}_2)$ was considered in [8]; q-deformations of the generalized KdV hierarchies were introduced [17].

In this paper we first formulate the results of [19] in terms of first order q-difference operators and q-gauge action. This naturally leads us to a generalization of the Drinfeld-Sokolov scheme to the setting of q-difference operators. The initial Poisson manifold is the loop group LSL_n of SL_n, or more generally, the loop group of a simply-connected simple Lie group G. Much of the needed Poisson formalism has already been developed by one of the authors in [29, 30]. Results of these works allow us to define a Poisson structure on the loop group, with respect to which the q-gauge action is Poisson. We then have to perform a reduction of this Poisson manifold with respect to the q-gauge action of the loop group LN of the unipotent subgroup N of G.

At this point we encounter a new kind of anomaly in the Poisson bracket relations, unfamiliar from the linear, (i.e., undeformed), situation. To describe it in physical terms, recall that the reduction procedure consists of two steps: (1) imposing the constraints and (2) passing to the quotient by the gauge group. An important point in the ordinary Drinfeld–Sokolov reduction is that these constraints are of first class, according to Dirac, i.e., their Poisson bracket vanishes on the constraint surface. In the q-difference case we have to choose carefully the classical r-matrix defining the initial Poisson structure on the loop group so as to make all constraints first class. If we use the standard r-matrix, some of the constraints are of second class, and so we have to modify the r-matrix.

In this paper we do that in the case of SL_2. We show that there is essentially a unique classical r-matrix compatible with the q-difference Drinfeld–Sokolov scheme. To the best of our knowledge, this classical r-matrix is new; it yields an elliptic deformation of the Lie bialgebra structure on the loop algebra of $\mathfrak{sl}_2$ associated with the Drinfeld “new” realization of quantized affine algebras [10, 22]. The result of the correspindingDrinfeld–Sokolov reduction is the q-deformation of the classical Virasoro algebra defined in [19].

We also construct a finite difference version of the Drinfeld–Sokolov reduction in the case of SL_2. This construction gives us a discrete version of the (classical) Virasoro algebra. We explain in detail the connection between our discrete Virasoro algebra and the lattice Virasoro algebra of Faddeev–Takhtajan–Volkov [34–36, 13]. We hope that our results will help to clarify further the meaning of the discrete Virasoro algebra and its relation to various integrable models.

The construction presented here can be generalized to the case of an arbitrary simply-connected simple Lie group. This is done in the second part of the paper [31] written by A. Sevostyanov and one of us.

The paper is arranged as follows. In Sect. 2 we recall the relevant facts of [11] and [19]. In Sect. 3 we interpret the results of [19] from the point of view of q-gauge transformations. Section 4 reviews some background material on Poisson structures on Lie groups following [29, 30]. In Sect. 5 we apply the results of Sect. 4 to the q-deformation of the Drinfeld–Sokolov reduction in the case of SL_2. In Sect. 6 we discuss the finite difference analogue of this reduction and compare its results with the Faddeev–Takhtajan–Volkov algebra.

2. Preliminaries

2.1. The differential Drinfeld–Sokolov reduction in the case of $\mathfrak{sl}_n$. Let $\mathcal{M}_n$ be the manifold of differential operators of the form

$$L = \partial^n + u_1(s)\partial^{n-2} + \ldots + u_{n-2}(s)\partial + u_{n-1}(s), \tag{2.1}$$

where $u_i(s) \in \mathbb{C}((s))$.

Adler [1] and Gelfand-Dickey [12] have defined a remarkable two-parameter family of Poisson structures on $\mathcal{M}_n$, with respect to which the corresponding KdV hierarchy is hamiltonian. In this paper we will only consider one of them, the so-called second bracket. There is a simple realization of this structure in terms of the Drinfeld–Sokolov reduction [11], Sect. 6.5. Let us briefly recall this realization.

Consider the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_n$ associated to $\mathfrak{sl}_n$; this is the central extension

$$0 \to \mathbb{C}K \to \widehat{\mathfrak{sl}}_n \to L\mathfrak{sl}_n \to 0,$$

see [20]. Let M_n be the hyperplane in the dual space to $\widehat{\mathfrak{sl}}_n$, which consists of linear functionals taking value 1 on K. Using the differential dt and the bilinear form tr AB on $\mathfrak{sl}_n$, we identify M_n with the manifold of first order differential operators

$$\partial_s + A(s), \qquad A(s) \in L\mathfrak{sl}_n.$$

The coadjoint action of the Lie group $\widehat{SL}_n$ on $\widehat{\mathfrak{sl}}_n^*$ factors through the loop group LSL_n and preserves the hyperplane M_n. The corresponding action of $g(s) \in LSL_n$ on M_n is given by

$$g(s) \cdot (\partial_s + A(s)) = g(s)(\partial_s + A(s))g(s)^{-1}, \tag{2.2}$$

or

$$A(s) \mapsto g(s)A(s)g(s)^{-1} - \partial_s g(s) \cdot g(s)^{-1}.$$

Consider now the submanifold M_n^J of M_n which consists of operators $\partial_s + A(s)$, where $A(s)$ is a traceless matrix of the form

$$\begin{pmatrix} * & * & * & \ldots & * & * \\ -1 & * & * & \ldots & * & * \\ 0 & -1 & * & \ldots & * & * \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & * & * \\ 0 & 0 & 0 & \ldots & -1 & * \end{pmatrix}. \tag{2.3}$$

To each element $\mathcal{L}$ of M_n^J one can naturally attach an n^{th} order scalar differential operator as follows. Consider the equation $\mathcal{L} \cdot \Psi = 0$, where

$$\Psi = \begin{pmatrix} \Psi_n \\ \Psi_{n-1} \\ \dots \\ \Psi_1 \end{pmatrix}.$$

Due to the special form (2.3) of $\mathcal{L}$, this equation is equivalent to an n^{th} order differential equation $L \cdot \Psi_1 = 0$, where L is of the form (2.1). Thus, we obtain a map $\pi : M_n^J \to \mathcal{M}_n$ sending $\mathcal{L}$ to L.

Let N be the subgroup of SL_n consisting of the upper triangular matrices, and LN be its loop group. If $g \in LN$ and Ψ is a solution of $\mathcal{L} \cdot \Psi = 0$, then $\Psi' = g\Psi$ is a solution of $\mathcal{L}' \cdot \Psi' = 0$, where $\mathcal{L}' = g\mathcal{L}g^{-1}$. But Ψ_1 does not change under the action of LN. Therefore $\pi(\mathcal{L}') = \pi(\mathcal{L})$, and we see that π factors through the quotient of M_n^J by the action of LN. The following proposition describes this quotient.

Proposition 1 ([11], Proposition 3.1). *The action of LN on M_n^J is free, and each orbit contains a unique operator of the form*

$$\partial_s + \begin{pmatrix} 0 & u_1 & u_2 & \dots & u_{n-2} & u_{n-1} \\ -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & -1 & 0 \end{pmatrix}. \tag{2.4}$$

But for $\mathcal{L}$ of the form (2.4), $\pi(\mathcal{L})$ is equal to the operator L given by formula (2.1). Thus, we have identified the map π with the quotient of M_n^J by LN and identified $\mathcal{M}_n$ with M_n^J/LN.

The quotient M_n^J/LN can actually be interpreted as the result of hamiltonian reduction. Denote by $\mathfrak{n}_+$ (resp., $\mathfrak{n}_-$) the upper (resp., lower) nilpotent subalgebra of $\mathfrak{sl}_n$; thus, $\mathfrak{n}_+$ is the Lie algebra of N.

The manifold M_n has a canonical Poisson structure, which is the restriction of the Lie-Poisson structure on $\widehat{\mathfrak{sl}}_n^*$ (such a structure exists on the dual space to any Lie algebra). The coadjoint action of LN on M_n is hamiltonian with respect to this structure. The corresponding moment map $\mu : M_n \to L\mathfrak{n}_- \simeq L\mathfrak{n}_+^*$ sends $\partial_s + A(s)$ to the lower-triangular part of $A(s)$. Consider the one-point orbit of LN,

$$J = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & -1 & 0 \end{pmatrix}.$$

Then $M_n^J = \mu^{-1}(J)$. Hence M_n is the result of hamiltonian reduction of M_n by LN with respect to the one-point orbit J.

The Lie-Poisson structure on M_n gives rise to a canonical Poisson structure on $\mathcal{M}_n$, which coincides with the second Adler-Gelfand-Dickey bracket, see [11], Sect. 6.5. The Poisson algebra of local functionals on $\mathcal{M}_n$ is called the classical $\mathcal{W}$-algebra associated to $\mathfrak{sl}_n$, and is denoted by $\mathcal{W}(\mathfrak{sl}_n)$.

Remark 1. For $\alpha \in \mathbb{C}$, let $M_{\alpha,n}$ be the hyperplane in the dual space to $\widehat{\mathfrak{sl}}_n$, which consists of linear functionals on $\widehat{\mathfrak{sl}}_n$ taking the value α on K. In the same way as above (for $\alpha = 1$) we identify $M_{\alpha,n}$ with the space of first order differential operators

$$\alpha\partial_s + A(s), \qquad A(s) \in L\mathfrak{sl}_n.$$

The coadjoint action is given by the formula

$$A(s) \mapsto g(s)A(s)g(s)^{-1} - \alpha\partial_s g(s) \cdot g(s)^{-1}.$$

The straightforward generalization of Proposition 1 is true for any $\alpha \in \mathbb{C}$. In particular, for $\alpha = 0$ we obtain a description of the orbits in M_n^J under the adjoint action of LN. This result is due to B. Kostant [23].

Drinfeld and Sokolov [11] gave a generalization of Proposition 1 when $\mathfrak{sl}_n$ is replaced by an arbitrary semisimple Lie algebra $\mathfrak{g}$. The special case of their result, corresponding to $\alpha = 0$, is also due to Kostant [23]. $\square$

The Drinfeld–Sokolov reduction can be summarized by the following diagram:

$$\begin{array}{ccc} M_n^J & \hookrightarrow & M_n \\ \downarrow & & \downarrow \\ \mathcal{M}_n = M_n^J/LN & \hookrightarrow & M_n/LN \end{array}$$

There are three essential properties of the Lie-Poisson structure on M_n that make the reduction work:

(i) The coadjoint action of LSL_n on M_n is hamiltonian with respect to this structure.
(ii) The subgroup LN of LSL_n is admissible in the sense that the space S of LN-invariant functionals on M_n is a Poisson subalgebra of the space of all functionals on M_n.
(iii) Denote by μ_{ij} the function on M_n, whose value at $\partial + A \in M_n$ equals the (i, j) entry of A. The ideal in S generated by $\mu_{ij} + \delta_{i-1,j}, i > j$, is a Poisson ideal.

We will generalize this picture to the q-difference case.

2.2. The Miura transformation. Let $\mathcal{F}_n$ be the manifold of differential operators of the form

$$\partial_s + \begin{pmatrix} v_1 & 0 & 0 & \dots & 0 & 0 \\ -1 & v_2 & 0 & \dots & 0 & 0 \\ 0 & -1 & v_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & v_{n-1} & 0 \\ 0 & 0 & 0 & \dots & -1 & v_n \end{pmatrix}, \qquad (2.5)$$

where $\sum_{i=1}^n v_i = 0$.

We have a map $\mathbf{m} : \mathcal{F}_n \to \mathcal{M}_n$, which is the composition of the embedding $\mathcal{F}_n \to M_n^J$ and the projection $\pi : M_n^J \to \mathcal{M}_n$.

Using the definition of π above, $\mathbf{m}$ can be described explicitly as follows: the image of the operator (2.5) under $\mathbf{m}$ is the n^{th} order differential operator

$$\partial_s^n + u_1(s)\partial_s^{n-2} + \ldots + u_{n-1}(s) = (\partial_s + v_1(s)) \ldots (\partial_s + v_n(s)).$$

The map $\mathbf{m}$ is called the Miura transformation.

We want to describe the Poisson structure on $\mathcal{F}_n$ with respect to which the Miura transformation is Poisson. To this end, let us consider the restriction of the gauge action (2.2) to the opposite triangular subgroup LN_-; let $\overline{\mu} : M_n \to L\mathfrak{n}_+ \simeq L\mathfrak{n}_-^*$ be the corresponding moment map. The manifold $\mathcal{F}_n$ is the intersection of two level surfaces, $\mathcal{F}_n = \mu^{-1}(J) \cap \overline{\mu}^{-1}(0)$. It is easy to see that it gives a local cross-section for both actions (in other words, the orbits of LN and LN_- are transversal to $\mathcal{F}_n$). Hence $\mathcal{F}_n$ simultaneously provides a local model for the reduced spaces $\mathcal{M}_n = \mu^{-1}(J)/LN$ and $\overline{\mu}^{-1}(0)/LN_-$. The Poisson bracket on $\mathcal{F}_n$ that we need to define is the so-called Dirac bracket (see, e.g., [16]), where we may regard the matrix coefficients of $\overline{\mu}$ as subsidiary conditions, which fix the local gauge. The computation of the Dirac bracket for the diagonal matrix coefficients v_i is very simple, since their Poisson brackets with the matrix coefficients of $\overline{\mu}$ all vanish on $\mathcal{F}_n$. The only correction arises due to the constraint $\sum_{i=1}^n v_i = 0$.

Denote by $v_{i,m}$ the linear functional on $\mathcal{F}_n$, whose value on the operator (2.5) is the m^{th} Fourier coefficient of $v_i(s)$. We obtain the following formula for the Dirac bracket on $\mathcal{F}_n$:

$$\{v_{i,m}, v_{i,k}\} = \frac{n-1}{n} m\delta_{m,-k},$$
$$\{v_{i,m}, v_{j,k}\} = -\frac{1}{n} m\delta_{m,-k}, \quad i < j.$$

Since $\mathcal{F}_n$ and $\mathcal{M}_n$ both are models of the same reduced space, we immediately obtain:

Proposition 2 ([11], Proposition 3.26). *With respect to this Poisson structure the map* $\mathbf{m} : \mathcal{F}_n \to \mathcal{M}_n$ *is Poisson.*

2.3. The q-deformations of $\mathcal{W}(\mathfrak{sl}_n)$ and Miura transformation. In this section we summarize relevant results of [19].

Let q be a non-zero complex number, such that $|q| < 1$. Consider the space $\mathcal{M}_{n,q}$ of q-difference operators of the form

$$L = D^n + t_1(s)D^{n-1} + \ldots + t_{n-1}(s)D + 1, \tag{2.6}$$

where $t_i(s) \in \mathbb{C}((s))$ for each $i = 1, \ldots, n$, and $[D \cdot f](s) = f(sq)$.

Denote by $t_{i,m}$ the functional on $\mathcal{M}_{n,q}$, whose value at L is the m^{th} Fourier coefficient of $t_i(s)$. Let $\mathcal{R}_{n,q}$ be the completion of the ring of polynomials in $t_{i,m}, i = 1, \ldots, N-1; m \in \mathbb{Z}$, which consists of finite linear combinations of expressions of the form

$$\sum_{m_1+\ldots+m_k=M} c(m_1, \ldots, m_k) \cdot t_{i_1,m_1} \ldots t_{i_k,m_k}, \tag{2.7}$$

where $c(m_1, \dots , m_k) \in \mathbb{C}$. Given an operator of the form (2.6), we can substitute the coefficients $t_{i,m}$ into an expression like (2.7) and get a number. Therefore elements of $\mathcal{R}_{n,q}$ define functionals on the space $\mathcal{M}_{n,q}$.

In order to define the Poisson structure on $\mathcal{M}_{n,q}$, it suffices to specify the Poisson brackets between the generators $t_{i,m}$. Let $T_i(z)$ be the generating series of the functionals $t_{i,m}$:

$$T_i(z) = \sum_{m\in\mathbb{Z}} t_{i,m} z^{-m}.$$

We define the Poisson brackets between $t_{i,m}$'s by the formulas [19]

$$\begin{aligned}\{T_i(z), T_j(w)\} = &\sum_{m\in\mathbb{Z}} \left(\frac{w}{z}\right)^m \frac{(1-q^{im})(1-q^{m(N-j)})}{1-q^{mN}} T_i(z)T_j(w)\\ &+ \sum_{r=1}^{\min(i,N-j)} \delta\left(\frac{wq^r}{z}\right) T_{i-r}(w)T_{j+r}(z) \\ &- \sum_{r=1}^{\min(i,N-j)} \delta\left(\frac{w}{zq^{j-i+r}}\right) T_{i-r}(z)T_{j+r}(w), \quad i \le j.\end{aligned} \tag{2.8}$$

In these formulas $\delta(x) = \sum_{m\in\mathbb{Z}} x^m$, and we use the convention that $t_0(z) \equiv 1$.

Remark 2. Note the difference between $t_i(s)$ and $T_i(z)$. The former is a Laurent power series, whose coefficients are *numbers*. The latter is a power series infinite in both directions, whose coefficients are *functionals* on $\mathcal{M}_{n,q}$. Thus, $T_i(z)$ is just the generating function for the functionals $t_{i,m}$. We use these generating functions merely to simplify our formulas for the Poisson brackets (so that we do not have to write a Poisson bracket between each individual pair $t_{i,m}$ and $t_{j,k}$).

Remark 3. The parameter q in formula (2.8) corresponds to q^{-2} in the notation of [19]. □

Now consider the space $\mathcal{F}_{n,q}$ of n-tuples of q-difference operators

$$(D + \lambda_1(s), \dots , D + \lambda_n(s)). \tag{2.9}$$

Denote by $\lambda_{i,m}$ the functional on $\mathcal{M}_{n,q}$, whose value is the m^{th} Fourier coefficient of $\lambda_i(s)$. We will denote by $\Lambda_i(z)$ the generating series of the functionals $\lambda_{i,m}$:

$$\Lambda_i(z) = \sum_{m\in\mathbb{Z}} \lambda_{i,m} z^{-m}.$$

We define a Poisson structure on $\mathcal{F}_{n,q}$ by the formulas [19]:

$$\{\Lambda_i(z), \Lambda_i(w)\} = \sum_{m\in\mathbb{Z}} \left(\frac{w}{z}\right)^m \frac{(1-q^m)(1-q^{m(N-1)})}{1-q^{mN}} \Lambda_i(z)\Lambda_i(w), \tag{2.10}$$

$$\{\Lambda_i(z), \Lambda_j(w)\} = -\sum_{m\in\mathbb{Z}} \left(\frac{wq^{N-1}}{z}\right)^m \frac{(1-q^m)^2}{1-q^{mN}} \Lambda_i(z)\Lambda_j(w), \quad i < j. \tag{2.11}$$

Now we define the q-deformation of the Miura transformation as the map $\mathbf{m}_q : \mathcal{F}_{n,q} \to \mathcal{M}_{n,q}$, which sends the n-tuple (2.9) to

$$L = (D + \lambda_1(s))(D + \lambda_2(sq^{-1}))\dots(D + \lambda_n(sq^{-n+1})), \tag{2.12}$$

i.e.

$$t_i(s) = \sum_{j_1<\dots<j_i} \lambda_{j_1}(s)\lambda_{j_2}(sq^{-1})\dots\lambda_{j_i}(sq^{-i+1}). \tag{2.13}$$

Proposition 3 ([19]). *The map* $\mathbf{m}_q$ *is Poisson.*

2.4. q*-deformation of the Virasoro algebra.* Here we specialize the formulas of the previous subsection to the case of $\mathfrak{sl}_2$ (we will omit the index 1 in these formulas). We have the following Poisson bracket on $T(z)$:

$$\{T(z), T(w)\} = \sum_{m\in\mathbb{Z}} \left(\frac{w}{z}\right)^m \frac{1-q^m}{1+q^m} T(z)T(w) + \delta\left(\frac{wq}{z}\right) - \delta\left(\frac{w}{zq}\right). \tag{2.14}$$

The q-deformed Miura transformation reads:

$$\Lambda(z) \mapsto T(z) = \Lambda(z) + \Lambda(zq)^{-1}. \tag{2.15}$$

The Poisson bracket on $\Lambda(z)$:

$$\{\Lambda(z), \Lambda(w)\} = \sum_{m\in\mathbb{Z}} \left(\frac{w}{z}\right)^m \frac{1-q^m}{1+q^m} \Lambda(z)\Lambda(w). \tag{2.16}$$

3. Connection with q-Gauge Transformations

In this section we present the results of [19] in the form of q-difference Drinfeld–Sokolov reduction.

3.1. Presentation via first order q*-difference operators.* By analogy with the differential case, it is natural to consider the manifold $M_{n,q}$ of first order difference operators $D + A(s)$, where $A(s)$ is an element of the loop group LSL_n of SL_n. The group LSL_n acts on this manifold by the q-gauge transformations

$$g(s) \cdot (D + A(s)) = g(sq)(D + A(s))g(s)^{-1}, \tag{3.1}$$

i.e. $g(s) \cdot A(s) = g(sq)A(s)g(s)^{-1}$.

Now we consider the submanifold $M^J_{n,q} \subset M_{n,q}$ which consists of operators $D + A(s)$, where $A(s)$ is of the form (2.3). It is preserved under the q-gauge action of the group LN.

In the same way as in the differential case, we define a map $\pi_q : M^J_{n,q} \to \mathcal{M}_{n,q}$, which sends each element of $M^J_{n,q}$ to an n^{th} order q-difference operator L of the form (2.6). It is clear that the map π_q factors through the quotient of $M^J_{n,q}$ by LN. Now we state the q-difference analogue of Proposition 1.

Lemma 1. *The action of LN on $M^J_{n,q}$ is free and each orbit contains a unique operator of the form*

$$D + \begin{pmatrix} t_1 & t_2 & t_3 & \dots & t_{n-1} & 1 \\ -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & -1 & 0 \end{pmatrix}. \tag{3.2}$$

Proof. The proof is an exercise in elementary matrix algebra. For $\alpha = 1, \dots, n$, denote by $M^\alpha_{n,q}$ the subset of matrices from $M^J_{n,q}$ satisfying the property that all entries in their rows $i = \alpha + 1, \dots, n$ are zero except for the $(i, i-1)$ entry that is equal to -1. We will prove that given $A(s) \in M^\alpha_{n,q}, \alpha > 1$, there exists $g(s) \in LN$, such that $g(sq)A(s)g(s)^{-1} \in M^{\alpha-1}_{n,q}$. Since the condition is vacuous for $\alpha = n$, i.e. $M^n_{n,q} = M^J_{n,q}$, this will imply that each LN-orbit in $M^J_{n,q}$ contains an element of the form (3.2).

To prove the statement for a given α, we will recursively eliminate all entries of the α^{th} row of $A(s)$ (except the $(\alpha, \alpha-1)$ entry), from right to left using elementary unipotent matrices. Denote by $E_{i,j}(x)$ the upper unipotent matrix whose only non-zero entry above the diagonal is the (i, j) entry equal to x. At the first step, we eliminate the (α, n) entry $A_{\alpha,n}$ of $A(s)$ by applying the q-gauge transformation (3.1) with $g(s) = E_{\alpha-1,n}(-A_{\alpha,n}(s))$. Then we obtain a new matrix $A'(s)$, which still belongs to $M^\alpha_{n,q}$, but whose (α, n) entry is equal to 0. Next, we apply the q-gauge transformation by $E_{\alpha-1,n-1}(-A'_{\alpha,n-1}(s))$ to eliminate the $(\alpha, n-1)$ entry of $A'(s)$, etc. It is clear that at each step we do not spoil the entries that have already been set to 0. The product of the elementary unipotent matrices constructed at each step gives us an element $g(s) \in LN$ with the desired property that $g(sq)A(s)g(s)^{-1} \in M^{\alpha-1}_{n,q}$.

To complete the proof, it suffices to remark that if $A(s)$ and $A'(s)$ are of the form (3.2), and $g(sq)A(s)g(s)^{-1} = A'(s)$ for some $g(s) \in LN$, then $A(s) = A'(s)$ and $g(s) = 1$. □

For $\mathcal{L}$ of the form (2.4), $\pi_q(\mathcal{L})$ equals the operator L given by formula (2.6). Thus, we have identified the map π_q with the quotient of $M^J_{n,q}$ by LN and $\mathcal{M}_{n,q}$ with $M^J_{n,q}/LN$.

Remark 4. In the same way as above we can prove the following more general statement. Let R be a ring with an automorphism τ. It rives rise to an automorphism of $SL_n(R)$ denoted by the same character. Define $M^J_{\tau,n}$ as the set of elements of $SL_n(R)$ of the form (2.3). Let the group $N(R)$ act on $M^J_{\tau,n}(R)$ by the formula $g \cdot A = (\tau \cdot g)Ag^{-1}$. Then this action of $N(R)$ is free, and the quotient is isomorphic to the set $\mathcal{M}^J_{\tau,n}(R)$ of elements of $SL_n(R)$ of the form (3.2) (i.e. each orbit contains a unique element of the form (3.2)). Note that the proof is not sensible to whether $\tau = \text{Id}$ or not.

When $\tau = \text{Id}$, this result is well-known. It gives the classical normal form of a linear operator. Moreover, in that case R. Steinberg has proved that the subset $\mathcal{M}^J_{\text{Id},n}(K)$ of $SL_n(K)$, where K is an algebraically closed field, is a cross-section of the collection of regular conjugacy classes in $SL_n(K)$ [33], Theorem 1.4. Steinberg defined an analogous cross-section for any simply-connected semisimple algebraic group [33]. His results can be viewed as group analogues of Kostant's results on semisimple Lie algebras [23] (cf. Remark 1). Steinberg's cross-section is used in the definition of the discrete Drinfeld–Sokolov reduction in the general semisimple case (see [31]).[1]

[1] We are indebted to B. Kostant for drawing our attention to [33]

3.2. Deformed Miura transformation via q-gauge action. Let us attach to each element of $\mathcal{F}_{n,q}$ the q-difference operator

$$\Lambda = D + \begin{pmatrix} \lambda_1(s) & 0 & \dots & 0 & 0 \\ -1 & \lambda_2(sq^{-1}) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{n-1}(sq^{-n+2}) & 0 \\ 0 & 0 & \dots & -1 & \lambda_n(sq^{-n+1}) \end{pmatrix}, \tag{3.3}$$

where $\prod_{i=1}^{n} \lambda_i(sq^{-i+1}) = 1$.

Let $\widetilde{\mathbf{m}}_q : \mathcal{F}_{n,q} \to \mathcal{M}_{n,q}$ be the composition of the embedding $\mathcal{F}_{n,q} \to M_{n,q}^J$ and $\pi_q : M_{n,q}^J \to M_{n,q}^J/LN \simeq \mathcal{M}_{n,q}$. Using the definition of π_q above, one easily finds that for Λ given by (3.3), $\widetilde{\mathbf{m}}_q(\Lambda)$ is the operator (3.2), where $t_i(s)$ is given by formula (2.13).

Therefore we obtain

Lemma 2. *The map $\widetilde{\mathbf{m}}_q$ coincides with the q-deformed Miura transformation $\mathbf{m}_q$.*

Remark 5. Let G be a simply-connected semisimple algebraic group over $\mathbb{C}$. Let V_i be the i^{th} fundamental representation of G (in the case $G = SL_n$, $V_i = \Lambda^i\mathbb{C}^n$), and $\chi_i : G \to \mathbb{C}$ be the corresponding character, $\chi_i(g) = \mathrm{Tr}(g, V_i)$. Define a map $p : G \to \mathbb{C}^n$ by the formula $p(g) = (\chi_1(g), \dots, \chi_n(g))$. By construction, p is constant on conjugacy classes. In the case $G = SL_n$ the map p has a cross-section $r : \mathbb{C}^n \to SL_n(\mathbb{C})$:

$$(a_1, \dots, a_n) \mapsto \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & 1 \\ -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & -1 & 0 \end{pmatrix}.$$

The composition $r \circ p$, restricted to $M_{n,1}^J$ coincides with the map π_1. Moreover, $\widetilde{\mathbf{m}}_1$ can be interpreted as the restriction of p to the subset of SL_n consisting of matrices of the form (3.3). Hence $\widetilde{\mathbf{m}}_1$ sends $(\lambda_1, \dots, \lambda_n)$ to the elementary symmetric polynomials

$$t_i = \sum_{j_1 < \dots < j_i} \lambda_{j_1}\lambda_{j_2}\dots\lambda_{j_i},$$

which are the characters of the fundamental representations of SL_n. As we mentioned above, Steinberg has defined an analogue of the cross-section r for an arbitrary simply-connected semisimple algebraic group [33].

Formula (2.13) expressing $t_i(z)$ in terms of the $\lambda_j(z)$'s can be thought of as a q-deformation of the character formula of the i^{th} fundamental representations of SL_n. It is interesting that the same interpretation is also suggested by the definition of $\mathcal{W}_q(\mathfrak{sl}_n)$ as the center of a completion of the quantized universal enveloping algebra $U_q(\widehat{\mathfrak{sl}}_n)_{-h^\vee}$ [19]. Namely, $t_i(z)$ is then defined as the (q-deformed) trace of the so-called L-operator acting on $\Lambda^i\mathbb{C}^n$ considered as a representation of $U_q(\widehat{\mathfrak{sl}}_n)$, see [26, 19] (note also that $t_i(z)$ is closely connected with a transfer-matrix of the corresponding integrable spin model). $\square$

Thus, we have now represented $\mathcal{M}_{n,q}$ as the quotient of the submanifold $M^J_{n,q}$ of the manifold $M_{n,q}$ of first order q-difference operators by the action of the group LN (acting by q-gauge transformations). We have also interpreted the q-deformed Miura transformation in these terms. In the next sections we discuss the Poisson structure on $M_{n,q}$, which gives rise to the Poisson structure on $\mathcal{M}_{n,q}$ given by explicit formula (2.8).

4. Poisson Structures

4.1. Overview. In view of the previous section, the following diagram is the q-difference analogue of the diagram presented at the end of Sect. 2.

$$
\begin{array}{ccc}
M^J_{n,q} & \hookrightarrow & M_{n,q} = (LG,\, \eta^q_*) \\
\downarrow & & \downarrow \\
\mathcal{M}_{n,q} = M^J_{n,q}/LN & \hookrightarrow & M_{n,q}/LN
\end{array}
$$

As in the differential case, in order to define a q-deformation of the Drinfeld–Sokolov reduction we need to find a Poisson structure η^q_* on $M_{n,q}$ and a Poisson-Lie structure η on LSL_n satisfying the following properties:

(i) The action $LSL_n \times M_{n,q} \to M_{n,q}$ by q-gauge transformations is Poisson.
(ii) The subgroup LN of LSL_n is admissible in the sense that the algebra S_q of LN-invariant functionals on $M_{n,q}$ is a Poisson subalgebra of the algebra of all functionals on $M_{n,q}$.
(iii) Denote by μ_{ij} the function on $M_{n,q}$, whose value at $D + A \in M_{n,q}$ equals the (i, j) entry of A. The ideal in S_q generated by $\mu_{ij} + \delta_{i-1,j}, i > j$, is a Poisson ideal.

Geometrically, the last condition means that $\mathcal{M}_{n,q}$ is a Poisson submanifold of the quotient $M_{n,q}/LN$.

For the sake of completeness, we recall the notions mentioned above. Let M be a Poisson manifold, and H be a Lie group, which is itself a Poisson manifold. An action of H on M is called Poisson if $H \times M \to M$ is a Poisson map (here we equip $H \times M$ with the product Poisson structure). In particular, if the multiplication map $H \times H \to H$ is Poisson, then H is called a Poisson-Lie group.

In this section we describe the general formalism concerning problems (i)–(iii) above. Then in the next section we specialize to $M_{2,q}$ and give an explicit solution of these problems.

4.2. Lie bialgebras. Let $\mathfrak{g}$ be a Lie algebra. Recall [9] that $\mathfrak{g}$ is called a Lie bialgebra, if $\mathfrak{g}^*$ also has a Lie algebra structure, such that the dual map $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ is a one-cocycle. We will consider *factorizable* Lie bialgebras $(\mathfrak{g}, \delta)$ satisfying the following conditions:

(1) There exists a linear map $r_+ : \mathfrak{g}^* \to \mathfrak{g}$, such that both r_+ and $r_- = -r_+^*$ are Lie algebra homomorphisms.
(2) The endomorphism $t = r_+ - r_-$ is $\mathfrak{g}$-equivariant and induces a linear isomorphism $\mathfrak{g}^* \to \mathfrak{g}$.

Instead of the linear operator $r_+ \in \mathrm{Hom}(\mathfrak{g}^*, \mathfrak{g})$ one often considers the corresponding element r of $\mathfrak{g}^{\otimes 2}$ (or a completion of $\mathfrak{g}^{\otimes 2}$ if $\mathfrak{g}$ is infinite-dimensional). The element r (or its image in the tensor square of a particular representation of $\mathfrak{g}$) is called a classical r-matrix. It satisfies the classical Yang-Baxter equation:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{4.1}$$

In terms or r, $\delta(x) = [r, x], \forall x \in \mathfrak{g}$ (here $[a \otimes b, x] = [a, x] \otimes b + a \otimes [b, x]$). The maps $r_\pm : \mathfrak{g}^* \to \mathfrak{g}$ are given by the formulas: $r_+(y) = (y \otimes \mathrm{id})(r), r_-(y) = -(\mathrm{id} \otimes y)(r)$.

Property (2) above means that $r + \sigma(r)$, where $\sigma(a \otimes b) = b \otimes a$ is a non-degenerate $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}^*$.

Set $\mathfrak{g}_\pm = \mathrm{Im}(r_\pm)$. Property (1) above implies that $\mathfrak{g}_\pm \subset \mathfrak{g}$ is a Lie subalgebra. The following statement is essentially contained in [5] (cf. also [28]).

Lemma 3. *Let* $(\mathfrak{g}, \mathfrak{g}^*)$ *be a factorizable Lie bialgebra. Then*

(1) The subspace $\mathfrak{n}_\pm = r_\pm(\mathrm{Ker}\, r_\mp)$ *is a Lie ideal in* $\mathfrak{g}_\pm$.
(2) The map $\theta : \mathfrak{g}_+/\mathfrak{n}_+ \to \mathfrak{g}_-/\mathfrak{n}_-$ *which sends the residue class of* $r_+(X), X \in \mathfrak{g}^*$, *modulo* $\mathfrak{n}_+$ *to that of* $r_-(X)$ *modulo* $\mathfrak{n}_-$ *is a well-defined isomorphism of Lie algebras.*
(3) Let $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ *be the direct sum of two copies of* $\mathfrak{g}$. *The map*

$$i : \mathfrak{g}^* \to \mathfrak{d}, \qquad X \mapsto (r_+(X), r_-(X))$$

is a Lie algebra embedding; its image $\mathfrak{g}^* \subset \mathfrak{d}$ *is*

$$\mathfrak{g}^* = \{(X_+, X_-) \in \mathfrak{g}_+ \oplus \mathfrak{g}_- \subset \mathfrak{d} | \overline{X}_- = \theta(\overline{X}_+)\},$$

where $\overline{Y}_\pm = Y \bmod \mathfrak{n}_\pm$.

Remark 6. The connection between our notation and that of [29] is as follows: the operator $r \in \mathrm{End}\, \mathfrak{g}$ of [29] coincides with the composition of $r_+ + r_-$ up to the isomorphism $t = r_+ - r_- : \mathfrak{g}^* \to \mathfrak{g}$; the bilinear form used in [29] is induced by t.

4.3. Poisson-Lie groups and gauge transformations. Let (G, η) (resp., (G^*, η^*)) be a Poisson-Lie group with factorizable tangent Lie bialgebra $(\mathfrak{g}, \delta)$ (resp., $(\mathfrak{g}^*, \delta^*)$). Let $G_\pm$ and $N_\pm$ be the Lie subgroups of G corresponding to the Lie subalgebras $\mathfrak{g}_\pm$ and $\mathfrak{n}_\pm$. We denote by the same symbol θ the isomorphism $G_+/N_+ \to G_-/N_-$ induced by $\theta : \mathfrak{g}_+/\mathfrak{n}_+ \to \mathfrak{g}_-/\mathfrak{n}_-$. Then the group G^* is isomorphic to

$$\{(g_+, g_-) \in G_+ \times G_- | \theta(\overline{g}_+) = \overline{g}_-\},$$

and we have a map $i : G^* \to G$ given by $i((g_+, g_-)) = g_+(g_-)^{-1}$.

Explicitly, the Poisson bracket on (G, η) can be written as follows:

$$\{\varphi, \psi\} = \langle r, \nabla\varphi \wedge \nabla\psi - \nabla'\varphi \wedge \nabla'\psi\rangle, \tag{4.2}$$

where for $x \in G$, $\nabla\varphi(x), \nabla'\varphi(x) \in \mathfrak{g}^*$ are defined by the formulas:

$$\langle \nabla\varphi(x), \xi\rangle = \frac{d}{dt}\varphi\left(e^{t\xi}x\right)|_{t=0}, \tag{4.3}$$

$$\langle \nabla'\varphi(x), \xi\rangle = \frac{d}{dt}\varphi\left(xe^{t\xi}\right)|_{t=0}, \tag{4.4}$$

for all $\xi \in \mathfrak{g}$. An analogous formula can be written for the Poisson bracket on (G^*, η^*). In formula (4.2) we use the standard notation $a \wedge b = (a \otimes b - b \otimes a)/2$.

By definition, the action of G on itself by left translations is a Poisson group action. There is another Poisson structure η_* on G which is covariant with respect to the adjoint action of G on itself and such that the map $i : (G^*, \eta^*) \to (G, \eta_*)$ is Poisson. It is given by the formula

$$\{\varphi, \psi\} = \langle r, \nabla\varphi \wedge \nabla\psi + \nabla'\varphi \wedge \nabla'\psi\rangle - \langle r, \nabla'\varphi \otimes \nabla\psi - \nabla'\psi \otimes \nabla\varphi\rangle. \tag{4.5}$$

Proposition 4. (1) *The map $i : G^* \to G$ is a Poisson map between the Poisson manifolds (G^*, η^*) and (G, η_*);*

(2) *The Poisson structure η_* on G is covariant with respect to the adjoint action, i.e. the map*

$$(G, \eta) \times (G, \eta_*) \to (G, \eta_*) : (g, h) \mapsto ghg^{-1}$$

is a Poisson map.

These results are proved in [29], § 3 (see also [30], § 2), using the notion of the Heisenberg double of G. Formula (4.5) can also be obtained directly from the explicit formulas for the Poisson structure η^* and for the embedding i.

More generally, let τ be an automorphism of G, such that the corresponding automorphism of $\mathfrak{g}$ satisfies $(\tau \otimes \tau)(r) = r$. Define a twisted Poisson structure η_*^τ on G by the formula

$$\begin{aligned}\{\varphi, \psi\} = &\langle r, \nabla\varphi \wedge \nabla\psi + \nabla'\varphi \wedge \nabla'\psi\rangle \\ &- \langle(\tau \otimes \mathrm{id})(r), \nabla'\varphi \otimes \nabla\psi - \nabla'\psi \otimes \nabla\varphi\rangle,\end{aligned} \tag{4.6}$$

and the twisted adjoint action of G on itself by the formula $g \cdot h = \tau(g)hg^{-1}$.

Theorem 1. *The Poisson structure η_*^τ on G is covariant with respect to the twisted adjoint action, i.e. the map*

$$(G, \eta) \times (G, \eta_*^\tau) \to (G, \eta_*^\tau) : (g, h) \mapsto \tau(g)hg^{-1}$$

is a Poisson map.

This result was proved in [29], § 3 (see also [30], § 2), using the notion of the twisted Heisenberg double of G. We will use Theorem 1 in two cases. In the first, G is the loop group of a finite-dimensional simple Lie group $\overline{G}$, and τ is the automorphism $g(s) \to g(sq), q \in \mathbb{C}^\times$. In the second, $G = \overline{G}^{\mathbb{Z}/N\mathbb{Z}}$, and τ is the automorphism $(\tau(g))_i \to g_{i+1}$. In the first case twisted conjugations coincide with q-gauge transformations, and in the second case they coincide with lattice gauge transformations.

4.4. Admissibility and constraints. Let M be a Poisson manifold, G a Poisson Lie group and $G \times M \to M$ be a Poisson action. A subgroup $H \subset G$ is called admissible if the space $C^\infty(M)^H$ of H-invariant functions on M is a Poisson subalgebra in the space $C^\infty(M)$ of all functions on M.

Proposition 5 ([29], Theorem 6). *Let $(\mathfrak{g}, \mathfrak{g}^*)$ be the tangent Lie bialgebra of G. A connected Lie subgroup $H \subset G$ with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ is admissible if $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ is a Lie subalgebra.*

In particular, G itself is admissible. Note that $H \subset G$ is a Poisson subgroup if and only if $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ is an ideal; in that case the tangent Lie bialgebra of H is $\left(\mathfrak{h}, \mathfrak{g}^*/\mathfrak{h}^\perp\right)$.

Let $H \subset G$ be an admissible subgroup, and I be a Poisson ideal in $C^\infty(M)^H$, i.e. I is an ideal in the ring $C^\infty(M)^H$, and $\{f, g\} \in C^\infty(M)^H$ for all $f \in I, g \in C^\infty(M)^H$. Then $C^\infty(M)^H/I$ is a Poisson algebra.

More geometrically, the Poisson structure on $C^\infty(M)^H/I$ can be described as follows. Assume that the quotient M/H exists as a smooth manifold. Then there exists a Poisson structure on M/H such that the canonical projection $\pi : M \to M/H$ is a Poisson map. Hamiltonian vector fields $\xi_\varphi, \varphi \in \pi^* C^\infty(M/H)$, generate an integrable distribution $\mathfrak{H}_\pi$ in TM. The following result is straightforward.

Lemma 4. *Let $V \subset M$ be a submanifold preserved by H. Then V/H is a Poisson submanifold of M/H if and only if V is an integral manifold of $\mathfrak{H}_\pi$.*

The integrality condition means precisely that the ideal I of all H-invariant functions on M vanishing on V is a Poisson ideal in $C^\infty(M)^H$, and that $C^\infty(V/H) = C^\infty(V)^H = C^\infty(M)^H/I$. If this property holds, we will say that the Poisson structure on M/H can be restricted to V/H.

$$\begin{array}{ccc} V & \hookrightarrow & M \\ \downarrow & & \downarrow \\ V/H & \hookrightarrow & M/H \end{array}$$

The Poisson structure on V/H can be described as follows. Let $N_V \subset T^*M\mid_V$ be the conormal bundle of V. Clearly, $T^*V \simeq T^*M\mid_V /N_V$. Let $\varphi, \psi \in C(V)^H$ and $\overline{d\varphi}, \overline{d\psi} \in T^*M\mid_V$ be any representatives of $d\varphi, d\psi \in T^*V$. Let $P_M \in \bigwedge^2 TM$ be the Poisson tensor on M.

Lemma 5. *We have*

$$\{\varphi, \psi\} = \left\langle P_M, \overline{d\varphi} \otimes \overline{d\psi} \right\rangle; \tag{4.7}$$

in particular, the right hand side does not depend on the choice of $\overline{d\varphi}, \overline{d\psi}$.

Remark 7. In the case of Hamiltonian action (i.e. when the Poisson structure on H is trivial), one can construct submanifolds V satisfying the condition of Lemma 4 using the moment map. Although a similar notion of the nonabelian moment map in the context of Poisson group theory is also available [24], it is less convenient. The reason is that the nonabelian moment map is "less functorial" than the ordinary moment map. Namely, if $G \times M \to M$ is a Hamiltonian action with moment map $\mu_G : M \to \mathfrak{g}^*$, its restriction to a subgroup $H \subset G$ is also Hamiltonian with moment $\mu_H = p \circ \mu_G$ (here $p : \mathfrak{g}^* \to \mathfrak{h}^*$ is the canonical projection). If G is a Poisson-Lie group, G^* its dual, $G \times M \to M$ a Poisson group action with moment $\mu_G : M \to G^*$, and $H \subset G$ a Poisson subgroup, the action of H still admits a moment map. But if $H \subset G$ is only admissible, then the restricted action does not usually have a moment map. This is precisely the case which is encountered in the study of the q-deformed Drinfeld–Sokolov reduction.

5. The q-Deformed Drinfeld–Sokolov Reduction in the Case of SL_2

In this section we apply the general results of the previous section to formulate a q-analogue of the Drinfeld–Sokolov reduction when $G = SL_2$.

5.1. Choice of r-matrix. Let $\mathfrak{g} = L\mathfrak{sl}_2$. We would like to define a factorizable Lie bialgebra structure on $\mathfrak{g}$ in such a way that the resulting Poisson-Lie structure η on LSL_2 and the Poisson structure η_*^q on $M_{2,q}$ satisfy the conditions (ii)–(iii) of Sect. 4.

Let $\{E, H, F\}$ be the standard basis in $\mathfrak{sl}_2$ and $\{E_n, H_n, F_n\}$ be the corresponding (topological) basis of $L\mathfrak{sl}_2 = \mathfrak{sl}_2 \otimes \mathbb{C}((s))$ (here for each $A \in \mathfrak{sl}_2$ we set $A_n = A \otimes s^n \in L\mathfrak{sl}_2$). Let τ be the automorphism of $L\mathfrak{sl}_2$ defined by the formula $\tau(A(s)) = A(sq)$ (we assume that q is generic). We have: $\tau \cdot A_n = q^n A_n$. To be able to use Theorem 1, the r-matrix $r \in L\mathfrak{sl}_2^{\otimes 2}$ defining the Lie bialgebra structure on $L\mathfrak{sl}_2$ has to satisfy the condition $(\tau \otimes \tau)(r) = r$. Hence the invariant bilinear form on $L\mathfrak{sl}_2$ defined by the symmetric part of r should also be τ-invariant.

The Lie algebra $L\mathfrak{sl}_2$ has a unique (up to a non-zero constant multiple) invariant non-degenerate bilinear form, which is invariant under τ. It is defined by the formulas

$$(E_n, F_m) = \delta_{n,-m}, \qquad (H_n, H_m) = 2\delta_{n,-m},$$

with all other pairings between the basis elements are 0. This fixes the symmetric part of the element r. Another condition on r is that the subgroup LN is admissible. According to Proposition 5, this means that $L\mathfrak{n}_+^\perp$ should be a Lie subalgebra of $L\mathfrak{sl}_2^*$.

A natural example of r satisfying these two conditions is given by the formula:

$$r_0 = \sum_{n\in\mathbb{Z}} E_n \otimes F_{-n} + \frac{1}{4} H_0 \otimes H_0 + \frac{1}{2} \sum_{n>0} H_n \otimes H_{-n}. \tag{5.1}$$

It is easy to verify that this element defines a factorizable Lie bialgebra structure on $\mathfrak{g}$. We remark that this Lie bialgebra structure gives rise to Drinfeld's "new" realization of the quantized enveloping algebra associated to $L\mathfrak{sl}_2$ [10, 22, 21]. As we will see in the next subsection, r_0 can not be used for the q-deformed Drinfeld–Sokolov reduction. However, the following crucial fact will enable us to perform the reduction. Let $L\mathfrak{h}$ be the loop algebra of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_2$.

Lemma 6. *For any $\rho \in \wedge^2 L\mathfrak{h}$, $r_0 + \rho$ defines a factorizable Lie bialgebra structure on $L\mathfrak{sl}_2$, such that $L\mathfrak{n}_+^\perp$ is a Lie subalgebra of $L\mathfrak{sl}_2^*$.*

The fact that $r_0 + \rho$ still satisfies the classical Yang-Baxter equation is a general property of factorizable r-matrices discovered in [5]. Lemma 6 allows us to consider the class of elements r given by the formula

$$r = \sum_{n\in\mathbb{Z}} E_n \otimes F_{-n} + \frac{1}{2} \sum_{m,n\in\mathbb{Z}} \phi_{n,m} \cdot H_n \otimes H_m, \tag{5.2}$$

where $\phi_{n,m} + \phi_{m,n} = \delta_{n,-m}$. The condition $(\tau \otimes \tau)(r) = r$ imposes the restriction $\phi_{n,m} = \phi_n \delta_{n,-m}$, so that (5.2) takes the form

$$r = \sum_{n\in\mathbb{Z}} E_n \otimes F_{-n} + \frac{1}{2} \sum_{n\in\mathbb{Z}} \phi_n \cdot H_n \otimes H_{-n}, \tag{5.3}$$

where $\phi_n + \phi_{-n} = 1$.

5.2. The reduction. Recall that $M_{2,q} = LSL_2 = SL_2((s))$ consists of the 2×2 matrices

$$M(s) = \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix}, \qquad ad - bc = 1. \tag{5.4}$$

We want to impose the constraint $c(s) = -1$, i.e. consider the submanifold $M^J_{2,q}$ and take its quotient by the (free) action of the group

$$LN = \left\{ \begin{pmatrix} 1 & x(s) \\ 0 & 1 \end{pmatrix} \right\}.$$

Let η be the Poisson-Lie structure on LSL_2 induced by r given by formula (5.3).

Let η^q_* be the Poisson structure on $M_{2,q}$ defined by formula (4.6), corresponding to the automorphism $\tau : g(s) \to g(sq)$. The following is an immediate corollary of Theorem 1, Proposition 5 and Lemma 6.

Proposition 6. *(1) The q-gauge action of (LSL_2, η) on $(M_{2,q}, \eta^q_*)$ given by formula $g(s) \cdot M(s) = g(sq)M(s)g(s)^{-1}$ is Poisson;*
(2) The subgroup $LN \subset LSL_2$ is admissible.

Thus, we have satisfied properties (i) and (ii) of Sect. 4. Now we have to choose the remaining free parameters ϕ_n so as to satisfy property (iii).

The Fourier coefficients of the matrix elements of the matrix $M(s)$ given by (5.4) define functions on $M_{2,q}$. We will use the notation a_m for the m^{th} Fourier coefficient of $a(s)$. Let $R_{2,q}$ be the completion of the ring of polynomials in $a_m, b_m, c_m, d_m, m \in \mathbb{Z}$, defined in the same way as the ring $\mathcal{R}_{n,q}$ of Sect. 2.3. Let $S_{2,q} \subset R_{2,q}$ be the subalgebra of LN-invariant functions. Denote by I be the ideal of $S_{2,q}$ generated by $\{c_n + \delta_{n,0}, n \in \mathbb{Z}\}$ (the defining ideal of $M^J_{2,q}$).

Property (iii) means that I is a Poisson ideal of $S_{2,q}$, which is equivalent to the condition that $\{c_n, c_m\} \in I$, i.e. that if $\{c_n, c_m\}$ vanishes on $M^J_{2,q}$. This condition means that the Poisson bracket of the constraint functions vanishes on the constraint surface, i.e. the constraints are of first class according to Dirac.

Let us compute the Poisson bracket between c_n's. First, we list the left and right gradients for the functions a_n, b_n, c_n, d_n (for this computation we only need the gradients of c_n's, but we will soon need other gradients as well). It will be convenient for us to identify $L\mathfrak{sl}_2$ with its dual using the bilinear form introduced in the previous section. Note that with respect to this bilinear form the dual basis elements to E_n, H_n, and F_n are $F_{-n}, H_{-n}/2$, and E_{-n}, respectively.

Explicit computation gives (for shorthand, we write a for $a(s)$, etc.):

$$\nabla a_m = s^{-m} \begin{pmatrix} \frac{1}{2}a & 0 \\ c & -\frac{1}{2}a \end{pmatrix}, \qquad \nabla b_m = s^{-m} \begin{pmatrix} \frac{1}{2}b & 0 \\ d & -\frac{1}{2}b \end{pmatrix},$$

$$\nabla c_m = s^{-m} \begin{pmatrix} -\frac{1}{2}c & a \\ 0 & \frac{1}{2}c \end{pmatrix}, \qquad \nabla d_m = s^{-m} \begin{pmatrix} -\frac{1}{2}d & b \\ 0 & \frac{1}{2}d \end{pmatrix},$$

$$\nabla' a_m = s^{-m} \begin{pmatrix} \frac{1}{2}a & b \\ 0 & -\frac{1}{2}a \end{pmatrix}, \qquad \nabla' b_m = s^{-m} \begin{pmatrix} -\frac{1}{2}b & 0 \\ a & \frac{1}{2}b \end{pmatrix},$$

$$\nabla' c_m = s^{-m} \begin{pmatrix} \frac{1}{2}c & d \\ 0 & -\frac{1}{2}c \end{pmatrix}, \qquad \nabla' d_m = s^{-m} \begin{pmatrix} -\frac{1}{2}d & 0 \\ c & \frac{1}{2}d \end{pmatrix}.$$

Now we can compute the Poisson bracket between c_n's using formula (4.6):

$$\{c_m, c_k\} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \left(\phi_n - \phi_{-n} + \phi_n q^n - \phi_{-n} q^{-n}\right) c_{-n+m} c_{n+k}. \tag{5.5}$$

Restricting to $M^J_{2,q}$, i.e. setting $c_n = -\delta_{n,0}$, we obtain:

$$\{c_m, c_k\}|_{M^J_{2,q}} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \left(\phi_m - \phi_{-m} + \phi_m q^m - \phi_{-m} q^{-m}\right) \delta_{m,-k}.$$

This gives us the following equation on ϕ_m's:

$$\phi_m - \phi_{-m} + \phi_m q^m - \phi_{-m} q^{-m} = 0.$$

Together with the previous condition $\phi_m + \phi_{-m} = 1$, this determines ϕ_m's uniquely:

Theorem 2. *The Poisson structure η^q_* satisfies property (iii) of the q-deformed Drinfeld-Sokolov reduction if and only if*

$$\phi_n = \frac{1}{1 + q^n}.$$

Consider the r-matrix (5.2) with $\phi_n = (1 + q^n)^{-1}$. For this r-matrix, the Lie algebras defined in section Sect. 4 are as follows: $\mathfrak{g}_\pm = L\mathfrak{b}_\mp, \mathfrak{n}_\pm = L\mathfrak{n}_\mp$, where $\mathfrak{n}_+ = \mathbb{C}E, \mathfrak{n}_- = \mathbb{C}F, \mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$. We have: $\mathfrak{g}_\pm/\mathfrak{n}_\pm \simeq L\mathfrak{h}$. The transformation θ on $L\mathfrak{h}$ induced by this r-matrix is equal to $-\tau$.

Explicitly, on the tensor product of the two 2-dimensional representations of $\mathfrak{sl}_2((t))$, the r-matrix looks as follows:

$$\begin{pmatrix} \phi\left(\frac{t}{s}\right) & 0 & 0 & 0 \\ 0 & -\phi\left(\frac{t}{s}\right) & \delta\left(\frac{t}{s}\right) & 0 \\ 0 & 0 & -\phi\left(\frac{t}{s}\right) & 0 \\ 0 & 0 & 0 & \phi\left(\frac{t}{s}\right) \end{pmatrix}, \tag{5.6}$$

where

$$\phi(x) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \frac{1}{1 + q^n} x^n.$$

Note that $2\pi\phi(xq^{1/2})$ coincides with the power series expansion of the Jacobi elliptic function dn (delta of amplitude).

Now we have satisfied all the necessary properties on the Poisson structures and hence can perform the q-Drinfeld–Sokolov reduction of Sect. 3 at the level of Poisson algebras. In the next subsection we check that it indeed gives us the Poisson bracket (2.14) on the reduced space $\mathcal{M}_{2,q} = M^J_{2,q}/LN$.

Remark 8. It is straightforward to identify the $q \to 1$ limit of the reduced Poisson algebra with the classical Virasoro algebra.

5.3. Explicit computation of the Poisson brackets. Introduce the generating series

$$A(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n},$$

and the same for other matrix elements of $M(s)$ given by formula (5.4). We fix the element r by setting $\phi_n = (1+q^n)^{-1}$ in formula (5.3) in accordance with Theorem 2. Denote

$$\varphi(z) = \sum_{n\in\mathbb{Z}}(\phi_n - \phi_{-n})z^n = \sum_{n\in\mathbb{Z}} \frac{1-q^n}{1+q^n} z^n. \tag{5.7}$$

Using the formulas for the gradients of the matrix elements given in the previous section and formula (4.6) for the Poisson bracket, we obtain the following explicit formulas for the Poisson brackets:

$$\begin{aligned}
&\{A(z), A(w)\} = \varphi\left(\frac{w}{z}\right) A(z)A(w),\\
&\{A(z), B(w)\} = -\delta\left(\frac{w}{z}\right) A(z)B(w),\\
&\{A(z), C(w)\} = \delta\left(\frac{w}{z}\right) A(z)C(w),\\
&\{A(z), D(w)\} = -\varphi\left(\frac{w}{z}\right) A(z)D(w),\\
&\{B(z), B(w)\} = 0,\\
&\{B(z), C(w)\} = \delta\left(\frac{w}{z}\right) A(z)D(w) - \delta\left(\frac{wq}{z}\right) A(z)A(w),\\
&\{B(z), D(w)\} = -\delta\left(\frac{wq}{z}\right) A(z)B(w),\\
&\{C(z), C(w)\} = 0,\\
&\{C(z), D(w)\} = \delta\left(\frac{w}{zq}\right) A(z)C(w),\\
&\{D(z), D(w)\} = \varphi\left(\frac{w}{z}\right) D(z)D(w) - \delta\left(\frac{wq}{z}\right) C(z)B(w) + \delta\left(\frac{w}{zq}\right) B(z)C(w).
\end{aligned}$$

Remark 9. The relations above can be presented in matrix form as follows. Let

$$L(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix},$$

and consider the operators $L_1 = L \otimes \mathrm{id}$, $L_2 = \mathrm{id} \otimes L$ acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$. The r-matrix (5.6) also acts on $\mathbb{C}^2 \otimes \mathbb{C}^2$. Formula (4.6) can be written as follows:

$$\begin{aligned}
\{L_1(z), L_2(w)\} = {}& \frac{1}{2} r^- \left(\frac{w}{z}\right) L_1(z)L_2(w) + \frac{1}{2} L_1(z)L_2(w) r^- \left(\frac{w}{z}\right)\\
&- L_1(z) r\left(\frac{wq}{z}\right) L_2(w) + L_2(w)\sigma(r)\left(\frac{zq}{w}\right) L_1(z),
\end{aligned}$$

where

$$r^{-}\left(\frac{w}{z}\right) = r\left(\frac{w}{z}\right) - \sigma(r)\left(\frac{z}{w}\right) = \begin{pmatrix} \frac{1}{2}\varphi\left(\frac{w}{z}\right) & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\varphi\left(\frac{w}{z}\right) & \delta\left(\frac{w}{z}\right) & 0 \\ 0 & -\delta\left(\frac{w}{z}\right) & -\frac{1}{2}\varphi\left(\frac{w}{z}\right) & 0 \\ 0 & 0 & 0 & \frac{1}{2}\varphi\left(\frac{w}{z}\right) \end{pmatrix}.$$

5.4. Reduced Poisson structure. We know that $\mathcal{M}_{2,q} = M^J_{2,q}/LN$ is isomorphic to

$$\left\{ \begin{pmatrix} t(s) & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

(see Sect. 3). The ring $\mathcal{R}_{2,q}$ of functionals on $\mathcal{M}_{2,q}$ is generated by the Fourier coefficients of $t(s)$. In order to compute the reduced Poisson bracket between them, we have to extend them to LN-invariant functions on the whole $M_{2,q}$. Set

$$\widetilde{t}(s) = a(s)c(sq) + d(sq)c(s). \tag{5.8}$$

It is easy to check that the Fourier coefficients $\widetilde{t}_m$ of $\widetilde{t}(s)$ are LN-invariant, and their restrictions to $M^J_{2,q}$ coincide with the corresponding Fourier coefficients of $t(s)$.

Let us compute the Poisson bracket between $\widetilde{t}_m$'s. Set

$$\widetilde{T}(z) = \sum_{m\in\mathbb{Z}} \widetilde{t}_m z^{-m}.$$

Using the explicit formulas above, we find

$$\begin{aligned} \{\widetilde{T}(z), \widetilde{T}(w)\} &= \varphi\left(\frac{w}{z}\right)\widetilde{T}(z)\widetilde{T}(w) \\ &+ \delta\left(\frac{wq}{z}\right)\Delta(z)c(w)c(wq^2) - \delta\left(\frac{w}{zq}\right)\Delta(w)c(z)c(zq^2), \end{aligned} \tag{5.9}$$

where $\Delta(z) = A(z)D(z) - B(z)C(z) = 1$. Hence, restricting to $M^J_{2,q}$ (i.e. setting $c(z) = 1$ in formula (5.9)), we obtain:

$$\{T(z), T(w)\} = \varphi\left(\frac{w}{z}\right)T(z)T(w) + \delta\left(\frac{wq}{z}\right) - \delta\left(\frac{w}{zq}\right).$$

This indeed coincides with formula (2.14).

Remark 10. Consider the subring $\widetilde{S}_{2,q}$ of the ring $R_{2,q}$, generated by $c_m, \widetilde{t}_m, m \in \mathbb{Z}$. The ring $\widetilde{S}_{2,q}$ consists of LN-invariant functionals on $M_{2,q}$, and hence it can serve as a substitute for the ring of functions on $M_{2,q}/LN$. Let us compute the Poisson brackets in $\widetilde{S}_{2,q}$. The Poisson brackets of $\widetilde{t}_m$'s are given by formula (5.9), and by construction $\{c_m, c_k\} = 0$. It is also easy to find that $\{c_m, \widetilde{t}_k\} = 0$. Hence $\widetilde{S}_{2,q}$ is a Poisson subalgebra of $R_{2,q}$. Thus, the q-deformed Drinfeld–Sokolov reduction can be interpreted as follows. The initial Poisson algebra is $R_{2,q}$. We consider its Poisson subalgebra $\widetilde{S}_{2,q}$ generated by c_m's and $\widetilde{t}_m$'s. The ideal I of $\widetilde{S}_{2,q}$ generated by $\{c_m + \delta_{m,0}, m \in \mathbb{Z}\}$ is a Poisson ideal. The quotient $\widetilde{S}_{2,q}/I$ is isomorphic to the q-Virasoro algebra $\mathcal{R}_{2,q}$ defined in Sect. 3.

5.5. q-deformation of Miura transformation. As was explained in Sect. 3.2, the q-Miura transformation of [19] is the map between two (local) cross-sections of the projection $\pi_q : M^J_{n,q} \to M^J_{n,q}/LN$. In the case of LSL_2, the first cross-section

$$\left\{ \begin{pmatrix} \lambda(s) & 0 \\ -1 & \lambda(s)^{-1} \end{pmatrix} \right\}$$

is defined by the subsidiary constraint $b(s) = 0$, and the second

$$\left\{ \begin{pmatrix} t(s) & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

is defined by the subsidiary constraint $d(s) = 0$. The map between them is given by the formula

$$\mathbf{m}_q : \lambda(s) \mapsto t(s) = \lambda(s) + \lambda(sq)^{-1}.$$

Now we want to recover formula (2.16) for the Poisson brackets between the Fourier coefficients λ_n of $\lambda(s)$, which makes the map $\mathbf{m}_q$ Poisson.

We have already computed the Poisson bracket on the second (canonical) cross-section from the point of view of Poisson reduction. Now we need to compute the Poisson bracket between the functions a_n's on the first cross-section, with respect to which the map $\mathbf{m}_q$ is Poisson. This computation is essentially similar to the one outlined in Sect. 3.2. The Poisson structure on the local cross-section is given by the Dirac bracket, which is determined by the choice of the subsidiary conditions, which fix the cross-section.

The Dirac bracket has the following property (see [16]). Suppose we are given constraints $\xi_n, n \in I$, and subsidiary conditions $\eta_n, n \in I$, on a Poisson manifold M, such that $\{\xi_k, \xi_l\} = \{\eta_k, \eta_l\} = 0, \forall k, l \in I$. Let f, g be two functions on M, such that $\{f, \xi_k\}$ and $\{g, \xi_k\}$ vanish on the common level surface of all ξ_k, η_k. Then the Dirac bracket of f and g coincides with their ordinary Poisson bracket.

In our case, the constraint functions are $c_m + \delta_{m,0}, m \in \mathbb{Z}$, and the subsidiary conditions are $b_m, m \in \mathbb{Z}$, which fix the local model of the reduced space. We have: $\{b_m, b_k\} = 0$, $\{c_m, c_k\} = 0$, and $\{a_m, b_k\} = 0$, if we set $b_m = 0, \forall m \in \mathbb{Z}$. Therefore we are in the situation described above, and the Dirac bracket between a_m and a_k coincides with their ordinary bracket. In terms of the generating function $A(z) = \sum_{m \in \mathbb{Z}} a_m z^{-m}$ it is given by the formula

$$\{A(z), A(w)\} = \varphi\left(\frac{w}{z}\right) A(z)A(w),$$

which coincides with formula (2.16). Thus, we have proved the Poisson property of the q-deformation of the Miura transformation from the point of view of the deformed Drinfeld–Sokolov reduction.

6. Lattice Virasoro Algebra

In this section we consider the lattice counterpart of the Drinfeld–Sokolov reduction. Our group is thus $\mathbf{G} = (SL_2)^{\mathbb{Z}/N\mathbb{Z}}$, where N is an integer, and τ is the automorphism of G, which maps (g_i) to (g_{i+1}). Poisson structures on $\mathbf{G}$ which are covariant with respect to lattice gauge transformations $x_n \mapsto g_{n+1} x_n g_n^{-1}$ have been studied already in [29] (cf.

also [2]). In order to make the reduction over the nilpotent subgroup $\mathbf{N} \subset \mathbf{G}$ feasible, we have to be careful in our choice of the r-matrix.

6.1. Discrete Drinfeld–Sokolov reduction. By analogy with the continuous case, we choose the element r defining the Lie bialgebra structure on $\mathfrak{g} = \mathfrak{sl}_2^{\oplus \mathbb{Z}/N\mathbb{Z}}$ as follows:

$$r = \sum_{n \in \mathbb{Z}/N\mathbb{Z}} E_n \otimes F_n + \frac{1}{4} \sum_{m,n \in \mathbb{Z}/N\mathbb{Z}} \phi_{n,m} H_n \otimes H_m,$$

where $\phi_{n,m} + \phi_{m,n} = 2\delta_{m,n}$. It is easy to see that r defines a factorizable Lie bialgebra structure on $\mathfrak{g}$. For Theorem 1 to be applicable, r has to satisfy the condition $(\tau \otimes \tau)(r) = r$, which implies that $\phi_{n,m} = \phi_{n-m}$.

An element of $\mathbf{G}$ is an N-tuple (g_i) of elements of SL_2:

$$g_k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}.$$

We consider $a_k, b_k, c_k, d_k, k \in \mathbb{Z}/N\mathbb{Z}$, as the generators of the ring of functions on $\mathbf{G}$.

The discrete analogue of the Drinfeld–Sokolov reduction consists of taking the quotient $\mathbf{M} = \mathbf{G}^J/\mathbf{N}$, where $\mathbf{G}^J = (G^J)^{\mathbb{Z}/N\mathbb{Z}}$,

$$G^J = \left\{ \begin{pmatrix} a & b \\ -1 & d \end{pmatrix} \right\},$$

and $\mathbf{N} = N^{\mathbb{Z}/N\mathbb{Z}}$, acting on G^J by the formula

$$(h_i) \cdot (g_i) = (h_{i+1} g_i h_i^{-1}). \tag{6.1}$$

It is easy to see that

$$\mathbf{M} \simeq \left\{ \begin{pmatrix} t_i & 1 \\ -1 & 0 \end{pmatrix}_{i \in \mathbb{Z}/N\mathbb{Z}} \right\}.$$

The element r with $\phi_{n,m} = \phi_{n-m}, \phi_k + \phi_{-k} = 2\delta_{k,0}$, defines a Lie bialgebra structure on $\mathfrak{g}$ and Poisson structures η, η_*^τ on $\mathbf{G}$. According to Theorem 1, the action of $(\mathbf{G}, \eta)$ on $(\mathbf{G}, \eta_*^\tau)$ given by formula (6.1) is Poisson.

As in the continuous case, for the Poisson structure η_*^τ to be compatible with the discrete Drinfeld–Sokolov reduction, we must have:

$$\{c_n, c_m\}|_{\mathbf{G}^J} = 0. \tag{6.2}$$

Explicit calculation analogous to the one made in the previous subsection shows that (6.2) holds if and only if

$$\phi_{n-1} + 2\phi_n + \phi_{n+1} = 2\delta_{n,0} + 2\delta_{n+1,0}.$$

The initial condition $\phi_0 = 1$ and periodicity condition give us a unique solution: for odd N, $\phi_k = (-1)^k$; for even N, $\phi_k = (-1)^k \left(1 - \frac{2k}{N}\right)$. In what follows we restrict ourselves to the case of odd N (note that in this case the linear operator $\mathrm{id} + \tau$ is invertible).

Continuing as in the previous subsection, we define

$$\widetilde{t}_n = a_n c_{n+1} + d_{n+1} c_n, \qquad n \in \mathbb{Z}/N\mathbb{Z}.$$

These are **N**-invariant functions on **G**. We find in the same way as in the continuous case:

$$\{\widetilde{t}_n, \widetilde{t}_m\} = \varphi_{n-m}\widetilde{t}_n\widetilde{t}_m + \delta_{n,m+1}c_m c_{m+2} - \delta_{n+1,m}c_n c_{n+2}, \tag{6.3}$$

$$\{\widetilde{t}_n, c_m\} = 0, \qquad \{c_n, c_m\} = 0,$$

where

$$\varphi_k = \frac{1}{2}(\phi_k - \phi_{-k}) = \begin{cases} 0, & k = 0, \\ (-1)^k, & k \neq 0. \end{cases}$$

The discrete Virasoro algebra $\mathbb{C}[t_i]_{i\in\mathbb{Z}/N\mathbb{Z}}$ is by definition the quotient of the Poisson algebra $\mathbb{C}[\widetilde{t}_i, c_i]_{i\in\mathbb{Z}/N\mathbb{Z}}$ by its Poisson ideal generated by c_i+1, $i \in \mathbb{Z}/N\mathbb{Z}$. From formula (6.3) we obtain the following Poisson bracket between the generators t_i:

$$\{t_n, t_m\} = \varphi_{n-m}t_n t_m + \delta_{n,m+1} - \delta_{n+1,m}. \tag{6.4}$$

The discrete Miura transformation is the map from the local cross-section

$$\left\{\begin{pmatrix} \lambda_n & 0 \\ -1 & \lambda_n^{-1} \end{pmatrix}\right\}$$

to **M**,

$$\lambda_n \mapsto t_n = \lambda_n + \lambda_{n+1}^{-1}. \tag{6.5}$$

It defines a Poisson map $\mathbb{C}[\lambda_i^{\pm}]_{i\in\mathbb{Z}/N\mathbb{Z}} \to \mathbb{C}[t_i]_{i\in\mathbb{Z}/N\mathbb{Z}}$, where the Poisson structure on the latter is given by the formula

$$\{\lambda_n, \lambda_m\} = \varphi_{n-m}\lambda_n\lambda_m. \tag{6.6}$$

Remark 11. The Poisson algebra $\mathbb{C}[t_i]_{i\in\mathbb{Z}/N\mathbb{Z}}$ can be considered as a regularized version of the q-deformed Virasoro algebra when $q = \epsilon$, where ϵ is a primitive N^{th} root of unity. Indeed, we can then consider $t(\epsilon^i)$, $i \in \mathbb{Z}/N\mathbb{Z}$, as generators and truncate in all power series appearing in the relations, summations over $\mathbb{Z}$ to summations over $\mathbb{Z}/N\mathbb{Z}$ divided by N. This means that we replace $\varphi(\epsilon^n)$ given by formula (5.7) by

$$\widetilde{\varphi}(\epsilon^n) = \frac{1}{N}\sum_{i\in\mathbb{Z}/N\mathbb{Z}} \frac{1-\epsilon^i}{1+\epsilon^i}\epsilon^{ni},$$

and $\delta(\epsilon^n)$ by $\delta_{n,0}$. The formula for the Poisson bracket then becomes:

$$\{t(\epsilon^n), t(\epsilon^m)\} = \widetilde{\phi}(\epsilon^{m-n})t(\epsilon^n)t(\epsilon^m) + \delta_{n,m+1} - \delta_{n+1,m}.$$

If we set $t(\epsilon^i) = t_i$, we recover the Poisson bracket (6.4), since it is easy to check that $\widetilde{\varphi}(\epsilon^{m-n}) = \varphi_{n-m}$.

One can apply the same procedure to the q-deformed $\mathcal{W}$-algebras associated to $\mathfrak{sl}_n$ and obtain lattice Poisson algebras. It would be interesting to see whether they are related to the lattice $\mathcal{W}$-algebras studied in the literature, e.g., in [6, 7]. In the case of $\mathfrak{sl}_2$, this connection is described in the next subsection.

6.2. Connection with Faddeev–Takhtajan–Volkov algebra. The Poisson structures (6.4) and (6.6) are nonlocal, i.e. the Poisson brackets between distant neighbors on the lattice are nonzero. However, one can define closely connected Poisson algebras possessing local Poisson brackets; these Poisson algebras can actually be identified with those studied by L. Faddeev, L. Takhtajan, and A. Volkov.

Let us first recall some results of [19] concerning the continuous case. As was explained in [19], one can associate a generating series of elements of the q-Virasoro algebra to an arbitrary finite-dimensional representation of $\mathfrak{sl}_2$. The series $T(z)$ considered in this paper corresponds to the two-dimensional representation. Let $T^{(2)}(z)$ be the series corresponding to the three-dimensional irreducible representation of $\mathfrak{sl}_2$. We have the following identity [19]:

$$T(z)T(zq) = T^{(2)}(z) + 1,$$

which can be taken as the definition of $T^{(2)}(z)$. From formula (2.15) we obtain:

$$\begin{aligned} T^{(2)}(z) &= \Lambda(z)\Lambda(zq) + \Lambda(z)\Lambda(zq^2)^{-1} + \Lambda(zq)^{-1}\Lambda(zq^2)^{-1} \\ &= A(z) + A(z)A(zq)^{-1} + A(zq)^{-1}, \end{aligned}$$

where

$$A(z) = \Lambda(z)\Lambda(zq) \tag{6.7}$$

(note that the series $A(z)$ was introduced in Sect. 7 of [19]). From formula (2.16) we find:

$$\{A(z), A(w)\} = \left(\delta\left(\frac{w}{zq}\right) - \delta\left(\frac{wq}{z}\right)\right) A(z)A(w).$$

It is also easy to find

$$\begin{aligned} \{T^{(2)}(z), T^{(2)}(w)\} = &\left(\delta\left(\frac{w}{zq}\right) - \delta\left(\frac{wq}{z}\right)\right)\left(T^{(2)}(z)T^{(2)}(w) - 1\right) \\ &+ \delta\left(\frac{wq^2}{z}\right) T(w)T(wq^3) - \delta\left(\frac{w}{zq^2}\right) T(z)T(zq^3). \end{aligned}$$

We can use the same idea in the lattice case. Let $\nu_n = \lambda_n\lambda_{n+1}$; this is the analogue of $A(z)$. We have:

$$\{\nu_n, \nu_m\} = (\delta_{n+1,m} - \delta_{n,m+1})\nu_n\nu_m, \tag{6.8}$$

and hence $\mathbb{C}[\nu_i^{\pm}]$ is a Poisson subalgebra of $\mathbb{C}[\lambda_n^{\pm}]$ with local Poisson brackets. We can also define $t_n^{(2)} = t_n t_{n+1} - 1$. The Poisson bracket of $t_n^{(2)}$'s is local:

$$\begin{aligned} \{t_n^{(2)}, t_m^{(2)}\} = &\left(\delta_{n+1,m} - \delta_{n,m+1}\right)\left(t_n^{(2)}t_m^{(2)} - 1\right) \\ &+ \delta_{n,m+2}t_m t_{m+3} - \delta_{n+2,m}t_n t_{n+3}. \end{aligned} \tag{6.9}$$

Unfortunately, it does not close on $t_n^{(2)}$'s, so that $\mathbb{C}[t_i^{(2)}]$ is not a Poisson subalgebra of $\mathbb{C}[t_i]$. But let us define formally

$$s_n = \frac{1}{1 + t_n^{(2)}} = t_n^{-1}t_{n+1}^{-1} = \frac{1}{(1+\nu_n)(1+\nu_{n+1}^{-1})}. \tag{6.10}$$

Then from formulas (6.10) and (6.8) we find:

$$\{s_n, s_m\} = s_n s_m \big((\delta_{n+1,m} - \delta_{n,m+1})(1 - s_n - s_m) - \\ - s_{n+1}\delta_{n+2,m} + s_{m+1}\delta_{n,m+2} \big). \tag{6.11}$$

Thus, the Poisson bracket closes among s_n's and defines a Poisson structure on $\mathbb{C}[s_i]_{i \in \mathbb{Z}/N\mathbb{Z}}$.

The Poisson algebra $\mathbb{C}[s_i]_{i \in \mathbb{Z}/N\mathbb{Z}}$ with Poisson bracket (6.11) was first introduced by Faddeev and Takhtajan in [34] (see formula (54)). We see that it is connected with our version of the discrete Virasoro algebra, $\mathbb{C}[t_i]$, by a change of variables (6.10). The Poisson algebra $\mathbb{C}[\nu_i^{\pm}]$ and the Poisson map $\mathbb{C}[\nu_i^{\pm}] \to \mathbb{C}[s_n]$ given by formula (6.10) were introduced by Volkov in [35] (see formulas (2) and (23)) following [34]; see also related papers [36, 13]. This map is connected with our version (6.5) of the discrete Miura transformation by a change of variables.

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References

1. Adler, M.: On a trace functional for formal pseudodifferential operators and the symplectic structure of the Korteweg–de Vries type equations. Invent. Math. **50**, 219–248 (1979)
2. Alekseev, A., Faddeev, L., Semenov-Tian-Shansky, M.: Hidden quantum group inside Kac–Moody algebras. Commun. Math. Phys. **149**, 335–345 (1992)
3. Asai, Y., Jimbo, M., Miwa, T., Pugai, Ya.: Bosonization of vertex operators for the $A_{n-1}^{(1)}$ vertex models. J. Phys. **A29**, 6595–6616 (1996)
4. Awata, H., Kubo, H., Odake, S., Shiraishi, J.: Quantum W_n algebras and Macdonald polynomials Comm. Math. Phys. **179**, 401–416 (1996)
5. Belavin, A.A., Drinfeld V.G.: Solutions of the classical Yang-Baxter equation for simple Lie algebras. Funct. Anal. Appl. **16**, 159–80 (1981)
6. Belov, A.D., Chaltikian, K.D.: Lattice analogues of W-algebras and classical integrable equations Phys. Lett., 268–274 (1993)
7. Bonora, L., Colatto, L.P., Constantinidis, C.P.: Toda lattice field theories, discrete W-algebras, Toda lattice hierarchies and quantum groups. Phys. Lett. **B387**, 759 (1996)
8. Ding, X.-M., Hou, B.-Y., Zhao, L.: $\hbar$-(Yangian) deformation of Miura map and Virasoro algebra. Preprint q-alg/9701014
9. Drinfeld, V.G.: Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang–Baxter equation. Sov. Math. Dokl. **27**, 68–71 (1983)
10. Drinfeld, V.G.: A new realization of Yangians and quantized affine algebras. Sov. Math. Dokl. **36**, 212–216 (1988)
11. Drinfeld, V.G., Sokolov, V.V.: Lie algebras and equations of Korteweg-de Vries type. Sov. Math. Dokl. **23**, 457–462 (1981); J. Sov. Math. **30**, 1975–2035 (1985)
12. Gelfand, I.M., Dickey, L.A.: *Family of Hamiltonian structures connected with integrable nonlinear equations*. Collected papers of I.M. Gelfand, vol. **1**, Berlin–Heidelberg–New York: Springer-Verlag, 1987, pp. 625–646
13. Faddeev, L.D., Volkov, A.Yu.: Abelian current algebras and the Virasoro algebra on the lattice. Phys. Lett. **B315**, 311–8 (1993)
14. Feigin, B., Frenkel, E.: Affine Lie algebras at the critical level and Gelfand-Dikii algebras. Int. J. Math. Phys. **A7**, suppl. A1, 197–215 (1992)

15. Feigin, B., Frenkel, E.: Quantum $\mathcal{W}$-algebras and elliptic algebras. Comm. Math. Phys. **178**, 653–678 (1996); q-alg/9508009
16. Flato, M., Lichnerowicz, A., Sternheimer, D.: Deformation of Poisson brackets, Dirac brackets and applications. J. Math. Phys. **17**, 1754 (1976)
17. Frenkel, E.: Deformations of the KdV hierarchy and related soliton equations, Int. Math. Res. Notices **2**, 55–76 (1996); q-alg/9511003
18. Frenkel, E.: Affine Kac-Moody algebras at the critical level and quantum Drinfeld–Sokolov reduction. PhD Thesis, Harvard University, 1991
19. Frenkel, E., Reshetikhin, N.: Quantum affine algebras and deformations of the Virasoro algebra and $\mathcal{W}$-algebras Comm. Math. Phys. **178**, 237–264 (1996); q-alg/9505025
20. Kac, V.: *Infinite-dimensional Lie algebras*. Third edition, Cambridge: Cambridge University Press, 1990
21. Kedem, R.: Singular R-matrices and Drinfeld's comultiplication. Preprint q-alg/9611001
22. Khoroshkin, S.M., Tolstoy, V.N.: On Drinfeld's realization of quantum affine algebras. J. Geom. Phys. **11**, 445–452 (1993)
23. Kostant, B.: On Whittaker vectors and representation theory. Invent. Math. **48**, 101–184 (1978)
24. Lu, J.H.: Momentum mapping and reduction of Poisson actions. In: *Symplectic geometry, groupoids and integrable systems*, Berkeley, 1989. P.Dazord and A.Weinstein (eds.), Berlin–Heidelberg–New York: Springer-Verlag, pp. 209–226
25. Lukyanov, S., Pugai, Ya.: Multi-point local height probabilities in the integrable RSOS models. Nucl. Phys. **B473**, 631 (1996)
26. Reshetikhin, N.Yu., Semenov-Tian-Shansky, M.A.: Central extensions of quantum current groups. Lett. Math. Phys. **19**, 133–142 (1990)
27. Reshetikhin, N.Yu., Semenov-Tian-Shansky, M.A.: Quantum R-matrices and factorization problems. In: *Geometry and Physics, essays in honor of I.M.Gelfand*. S.Gindikin and I.M.Singer (eds.), pp. 533-550. Amsterdam–London–New York: North Holland, 1991
28. Semenov-Tian-Shansky, M.A.: What is a classical r-matrix. Funct. Anal. Appl. **17**, 17–33 (1983)
29. Semenov-Tian-Shansky M.A.: Dressing action transformations and Poisson–Lie group actions. *Publ. RIMS.* **21**, 1237–1260 (1985)
30. Semenov-Tian-Shansky, M.A.: Poisson Lie groups, quantum duality principle and the quantum double. Contemporary Math. **175**, 219–248
31. Semenov-Tian-Shansky M.A., Sevostyanov A.V.: Drinfeld–Sokolov reduction for difference operators and deformations of $\mathcal{W}$-algebras II. General semisimple case. Preprint q-alg/9702016
32. Shiraishi, J., Kubo, H., Awata, H., Odake, S.: A quantum deformation of the Virasoro algebra and the Macdonald symmetric functions Lett. Math. Phys. **38**, 33–51 (1996)
33. Steinberg, R.: Regular elements of semisimple algebraic Lie groups, Publ. Math. I.H.E.S., **25**, 49–80 (1965)
34. Takhtajan, L.A., Faddeev, L.D.. Liouville model on the lattice, Lect. Notes in Phys. **246**, 166–179 (1986)
35. Volkov, A.Yu.: Miura transformation on a lattice. Theor. Math. Phys. **74**, 96–99 (1988)
36. Volkov, A.Yu.: Quantum Volterra model. Preprint HU-TFT-92-6 (1992)
37. Weinstein, A.: Local structure of Poisson manifolds, J. Different. Geom. **18**, 523–558 (1983)

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