Second part of the notes for 'Physics for W-algebraists'

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Abstract

These are notes of the seminar given by myself on physics for W-algebraists, in the team seminar of Pr. Suh UhiRinn at Seoul National University in May 2024. Following the first part of my seminar, I first introduce rational conformal field theories, and derive the WZW-model from the Sigma model, one of the biggest exemple of RCFT. I also compute its algebra, which is an affine Lie algebra. Then, I more formally introduce $\mathcal{W}$-algebras, give the exemple of Casimir algebras, and then use this intuition to derive the more general class of the quantum affine $\mathcal{W}$-algebras known to mathematicians. Be aware that these notes weren't proofread by anyone, and that there are most probably errors in it. One should not have too much faith in the details of every formula. However, the intuition behind each of them should be right.

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Chapter 1

$\mathcal{W}$-algebras

This section aims at introducing $\mathcal{W}$-algebras from a physical perspective, and at giving a quick overview of the general theory of $\mathcal{W}$-algebras on the physical side.

1.1 Minimal models

Before we start discussing $\mathcal{W}$-algebras, we should go to the basics to understand their origin and importance. Let's consider a 2D conformal field theory. We recall from the last notes that the Hilbert space of the theory takes the form

$$\mathcal{H} = \bigoplus_{a,b} V(h_a, c) \otimes \bar{V}(\bar{h}_b, c) \tag{1.1}$$

where a indexes the holomorphic dimensions of primary fields, and b indexes the antiholomorphic dimensions.

One of the important objects of a conformal field theory is its partition function, which allows one to compute correlation function among other things. In particular, if the theory is defined on a torus (if we assume we can compactify and loop around time), we can read the partition function of the theory using the formula:

$$Z(\tau) = \sum_{a,b} N_{ab} \chi(h_a, c) \bar{\chi}(\bar{h}_b, c) \tag{1.2}$$

Where, τ is the moduli of the torus, $\chi(h_a, c)$ is the character associated to $V(h_a, c)$, $\bar{\chi}(\bar{h}_b, c)$ is the character associated to $\bar{V}(\bar{h}_b, c)$, and N_{ab} is the degeneracy of representations with conformal dimension $(h_a, \bar{h}_b)$ in the theory.

Now let suppose we are a physicist wanting to classify 2D conformal field theories. Classifying all of them would be a very hard task, still unsolved today. We must therefore restreign ourselves to a smaller class of "simple" conformal field theories. How to choose which CFT is "simple enough" and which is not? In view of (1.2), we might want to study theories where the rank of the matrix N_{ab} is finite, or alternatively where the sum in (1.2) is finite. This is what was originally called rational field theories.

Indeed, the study of a theory gets simpler when the sum on the left is finite. Knowing the correlation function of the primary fields of a theory is enough to know the correlation function of all fields of the theory. But in a RCFT, this number is finite, such that one has a finite number of correlation functions to determine.

Moreover, assuming modularity, this condition of finiteness already puts heavy constraints on conformal field theories. It has been shown by Moore and Anderson that in a rational conformal field theory, the central charge c and the conformal dimensions h_a of the primary fields are all rational numbers, hence the name "rational".

For the sum in (1.2) to be finite, there should be a finite number of Virasoro primary fields allowed in the theory, generating a finite number of Verma modules. This condition is known to be realized when $c < 1$. Indeed, when $c < 1$, there exist only a finite number of conformal dimensions h such that the Verma module $V(h, c)$ is unitary. This discrete set of numbers is given by the Kac determinant. Therefore, unitary 2D CFTs with $c < 1$ are rational. They have been entirely classified, and are called *minimal model.*

However, the inverse happens for $c \geq 1$. It has been shown that in any 2D CFT where $c \geq 1$, there always exists an infinite number of Virasoro primary fields, generating an infinite number of Verma modules. 2D CFTs with $c = 1$ have been entirely classified [Flo93], as in the case of the minimal models. But what about theories with $c > 1$?

1.2 Rational conformal field theories

All conformal field theories with $c > 1$ have an infinite number of Verma modules. Is this the end of the discussion? Not necessarily. The solution lies in extending the Virasoro algebra. To see this, let's study the structure of a minimal model by changing perspective.

Let us consider a minimal model. Its Hilbert space takes the form

$$\mathcal{H} = \sum_{a=1}^{N_a} \sum_{b=1}^{N_b} V(h_a, c) \otimes \bar{V}(\bar{h}_b, c) \tag{1.3}$$

From the operator perspective, this can be written as

$$\mathcal{H} = \sum_{a=1}^{N_a} \sum_{b=1}^{N_b} [\varphi_{h_a}] \otimes [\bar{\varphi}_{h_a}] \tag{1.4}$$

Where $[\varphi_{h_a}]$ is the family of operators descending from the primary field φ_{h_a}, or in other words is the operator equivalent of $V(h_a, c)$.

Let's focus on the holomorphic dimension. Let $\varphi_h, \varphi_{h'}$ two primary fields. The conformal field theory is closed under the operator product expansion, and it only contains a finite number of conformal families. Moreover, we know that the OPE of two fields in each of the families will be similar to the OPE of the two primary fields, modulo the appearance of lowering operators from the Virasoro algebra. Therefore, we can generally decompose the operator product expansion of members of the two families by counting the number of appearance of operators from other families, as

$$[\varphi_h] \times [\varphi_{h'}] = \sum_a N^a_{hh'} [\varphi_a] \tag{1.5}$$

This is called the fusion rule.

Among the different holomorphic conformal families, only one has conformal weight $h = 0$. Indeed, writing the field generating this module φ_0, this field is a Virasoro primary field of conformal dimensions 0. Therefore, it is constant under any local conformal transformation. We can then assume it to be the identity. We call this module the (holomorphic) chiral algebra of the theory, and write it $\mathcal{A}$. Since $\mathcal{A}$ contains the identity, we can determine its fusion rule based of the OPE of the identity. We have

$$N^a_{b0} = N^a_{0b} = \delta^a_b \tag{1.6}$$

This shows that the OPE gives an action of $\mathcal{A}$ onto all of the holomorphic conformal families, or alternatively that all holomorphic Verma modules are representations of $\mathcal{A}$. A theory is then a rational conformal field theory in the sense given above when both $\mathcal{A}$ and the antiholomorphic chiral algebra $\bar{\mathcal{A}}$ have a finite number of possible representations.

This discussion is actually quite trivial, as $\mathcal{A}$ is the conformal family generated by the Virasoro algebra from the identity and is therefore nothing more than the (holomorphic) Virasoro algebra itself. We are just saying that all Verma modules are representations of the Virasoro algebra.

However, one can generalize such discussion by extending $\mathcal{A}$ and $\bar{\mathcal{A}}$ into bigger algebras. Focusing once again on the holomorphic dimension, let us consider a conformal field theory where the Virasoro algebra is embedded into a bigger algebra $\mathcal{A}$ which can act on every state of the system. Instead of decomposing the Hilbert space into highest weight representations of the Virasoro algebra, one can decompose the Hilbert space into highest weight representations of $\mathcal{A}$, where the highest weight states are primary with respect to $\mathcal{A}$, or alternatively where the generating fields are primary with respect to $\mathcal{A}$.

In this case, on a torus, one can once again write the partition function

$$Z(\tau) = \sum_{a,b} N_{ab} \chi(h_a, c) \bar{\chi}(\bar{h}_b, c) \tag{1.7}$$

where the $\chi(h_a, c)$ are instead characters of the holomorphic chiral algebra, and similarly in the antiholomorphic dimension. We then define generally a *rational conformal field theory* (RCFT) as a theory where the rank of the newly defined matrix N_{ab} is finite, or alternatively where the sum on the right hand side is finite. Equivalently, we can say that a CFT is rational if its Hilbert space decomposes into a finite sum of representations of the chiral algebras $\mathcal{A}$ and $\bar{\mathcal{A}}$

$$\mathcal{H} = \sum_{i=1}^{N_i} \sum_{j=1}^{N_j} \mathcal{H}_i \otimes \bar{\mathcal{H}}_j \tag{1.8}$$

Once again, we see that RCFTs are easier to study than general CFTs. We can still determine the correlation function of any field from the correlation function of general primary fields (with respect to the chiral algebra), of which there is a finite number. Thus, the complete understanding of a RCFT once again reduces to the computation of a finite number of correlation functions.

Classifying rational conformal field theories comes down to classifying all acceptable, non trivial extensions of the Virasoro algebra which admits only a finite number of unitary representations. However, a lot of questions on this subject are still open up to now. We know that additional symmetries in the theory can in general help extend the Virasoro algebra. A construction of such extension under some conditions is detailed in [Int90], for the interested reader.

1.3 Holomorphic CFTs

RCFTs are a very important class of CFTs, playing a huge role in the study of string theory among other theories. A lot of theories naturally appearing are RCFTs, without being minimal models. In particular, string theory requires RCFTs with $c > 1$, as do phase transition systems of second order. But we saw just before that the classification of RCFTs can be reduced to that of chiral algebras $\mathcal{A}$ extending the Virasoro algebra and having a finite number of representations.

We would therefore like to study in more details the possible extensions of the Virasoro algebra, and more generally the identity sector of RCFTs (the space of operators corresponding to the chiral algebra). As the identity sector of a RCFT is just generated by the two chiral algebras acting on the identity operator, this sector is closed under OPE. It therefore makes a completely consistent theory by itself. Furthermore, in such a theory, both chiral parts are completely decorrelated. As such, one could study only one side of the theory, which is also completely consistent by itself.

Definition 1.3.1. An *holomorphic field theory* is a conformal field theory where the Hilbert space is reduced to its (without loss of generality, holomorphic) chiral algebra. It can also be equivalently defined as a rational field theory whose fusion rule is trivial, highlighting the motivation behind the study of such a theory.

However, we want to abstract ourselves from physics. What truly is the object behind a holomorphic conformal field theory? We should have an Hilbert space $\mathcal{H}$ corresponding to the space of states, and a map from the space of states to the space of fields $|\varphi\rangle \to V(|\varphi\rangle, z)$ corresponding to the state-operator correspondance. Moreover, the space of fields should contain the stress-energy tensor $V(|L\rangle, z) = T(z)$ such that writing

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \tag{1.9}$$

the family $(L_n)_n$ satisfies the Virasoro algebra. The space of states should also contain the vacuum $|0\rangle$, which is the unique state such that

$$\forall|\varphi\rangle \in \mathcal{H}, \quad V(|\varphi\rangle, z)|0\rangle = e^{zL_{-1}}|\varphi\rangle \tag{1.10}$$

Finally, the operators should act well on the states, to create nice correlation functions. For all $|\varphi_1\rangle, |\varphi_2\rangle, |\varphi_3\rangle, |\varphi_4\rangle \in \mathcal{H}$, we should have that

- $\langle\varphi_1|V(|\varphi_2\rangle, z)|\varphi_3\rangle$ is meromorphic in z
- $\langle\varphi_1|V(|\varphi_2\rangle, z)V(|\varphi_3\rangle, w)|\varphi_4\rangle$ is meromorphic in z and w for $|z| > |w|$
- $\langle\varphi_1|V(|\varphi_2\rangle, z)V(|\varphi_3\rangle, w)|\varphi_4\rangle = \varepsilon_{\varphi_2\varphi_3}\langle\varphi_1|V(|\varphi_3\rangle, w)V(|\varphi_2\rangle, z)|\varphi_4\rangle$ by analytic continuation, with $\varepsilon_{\varphi_2\varphi_3}$ depending wether $|\varphi_2\rangle$ and $|\varphi_3\rangle$ are "bosonic" or "fermionic".

These axioms imply a few things, which are expected from the physical point of view. First of all, the state-operator map is an isomorphism. Moreover, the operator L_0 is ad-diagonalizable on $\mathcal{H}$ and can only have integer or half-integer eigenvalues. This leads to the definition of bosonic or fermionic states, respectively as eigenstates of L_0 with integer or half-integer eigenvalue.

According to the grading implied by L_0, we write

$$\mathcal{H} = \bigoplus_h \mathcal{H}_h \tag{1.11}$$

Then, for $|\varphi\rangle \in \mathcal{H}_h$, we write

$$V(|\varphi\rangle, z) = \sum_n \varphi_n z^{-n-h} \tag{1.12}$$

where n is integer or half-integer depending on the integrity of h. With this notation, we notice that

$$\begin{aligned} \varphi_{-h}|0\rangle &= |\varphi\rangle \\ \forall n \geq -h+1, \quad \varphi_n|0\rangle &= 0 \end{aligned} \tag{1.13}$$

We can also rewrite the OPE as can usually be done in term of modes

$$\begin{aligned} A(z)B(w) = B(w)A(z) &= \sum_{r=r_0}^{\infty} (A_{(r)}B)(w)(z-w)^r \\ (A_{(r)}B)(w) &= V(A_{-r-h_A}B_{-h_B}|0\rangle, w) \end{aligned} \tag{1.14}$$

which gives at the same time

$$: A(z)B(w) := V(A_{-h_A}B_{-h_B}, z-w) \tag{1.15}$$

1.4 $\mathcal{W}$-algebras

We want to further restreign the structure of a holomorphic conformal field theory, and only consider "finite" holomorphic conformal field theories, for some notion of finiteness. Indeed, this is similar to considering RCFTs instead of general CFTs. The objective here is to study some structure that we can actually describe, and hope to classify.

What is the right notion of finiteness? As in the case of RCFTs where we only consider theories with a finite number of conformal families, that is theories with a finite number of primary fields (primary with regards to the chiral algebras), we here want the holomorphic theory to be finitely generated. The question is then: generated with regards to what operation? We should allow for the derivative and the normal-ordered product, as it is at the basis of vertex algebras. Since any holomorphic conformal field theory should also have an energy-momentum field, we can use the derivative to get all modes of the energy-momentum field and applying it to the generators, we get the descendants of the generator under the Virasoro algebra. We can therefore suppose the generators are Virasoro primary. If we were to also allow for the modes of the chiral algebra to act on the system, then any holomorphic CFT would be finite, generated by the vacuum only, by definition of the chiral algebra. As such, we understand that only allowing for the derivative and the action of the Virasoro algebra is the right way to define the notion of finiteness. We should consider holomorphic conformal field theories which are generated from a finite number of primary fields (and from the energy-momentum fieldy), using OPEs and derivatives. Looking at the original decomposition of the Hilbert space in Verma modules, this amounts to saying that the chiral algebra should be made up of finitely many Verma modules.

Definition 1.4.1. A *quantum $\mathcal{W}$-algebra* is an holomorphic conformal field theory such that the Hilbert space $\mathcal{H}$ is generated, under derivations and normal ordered product, by a finite set of states $(|i\rangle)_i$ including $|L\rangle$ whose corresponding fields $(W^{(s_i)}(z))$ are quasi-primary fields with integer conformal dimension s_i.

If $\mathcal{H}$ is a $\mathcal{W}$-algebra, it is then spanned by lexicographically ordered states of the form

$$W^{(s_{i_1})}_{-m_1-s_{i_1}} \dots W^{(s_{i_n})}_{-m_n-s_{i_n}}|0\rangle \tag{1.16}$$

We usually write the $\mathcal{W}$-algebra generated by n fields $(|i\rangle)_{i\in\mathbb{N}_n}$ of conformal dimensions $(h_i)_{i\in\mathbb{N}_n}$: $\mathcal{W}(h_1,\dots h_n)$. We notice that as the energy-momentum field is always present in the theory, and is always of conformal dimension 2. Therefore, there is always a 2 in the conformal dimensions $(h_i)_{i\in\mathbb{N}_n}$. In fact, the energy-momentum field is in general the only field which is not primary. Moreover, we recall that the conformal dimensions should all be integers, for reasons of modular invariance. The first known non-trivial exemple of $\mathcal{W}$-algebra was Zamalodchikov's $\mathcal{W}(2,3)$, usually written $\mathcal{W}_3$.

In the general case, we cannot construct a $\mathcal{W}$-algebra for any conformal dimension $(h_i)_{i\in\mathbb{N}_n}$. Finding the class of sequences of conformal dimensions $(h_i)_{i\in\mathbb{N}_n}$ such that $\mathcal{W}(h_1,\dots h_n)$ is well defined is still today an open problem. Moreover, in certain case, $\mathcal{W}(h_1,\dots h_n)$ can have multiple constructions leading to multiple different $\mathcal{W}$-algebras. Therefore, we should emphasize that this nomenclature can be ambiguous.

It is not too hard to compute all possible $\mathcal{W}$-algebras with a small number of primary fields. In particular, such extensive studies have been done for $\mathcal{W}$-algebras with 2 generating fields, of the form $\mathcal{W}(2,n)$ for $n\in\mathbb{N}^*$. However, it is still important to understand all of the possible $\mathcal{W}$-algebras, and not simply the smaller ones which can be computed by hand. Indeed, this is shown by the main result of [Eho+92], stating that any bosonic $\mathcal{W}$-algebra generated by k different fields and having a finite number of representations (as to generate a RCFT) must have its central charge obey $c < k$. We must consider bigger $\mathcal{W}$-algebras to consider theories with bigger central charges.

We end this subsection by discussing possible generalizations of this definition. First, we can also allow for the $\mathcal{W}$-algebra to have an infinite number of generating fields. Relaxing the definition this way allows one to consider algebras such as $\mathcal{W}_\infty$ or $\hat{\mathcal{W}}_{1,\infty}$, which are useful in the quest for a universal cover of $\mathcal{W}$-algebras. We can also relax the definition by allowing for the generators to have half-integer conformal dimensions. This allows one to define graded $\mathcal{W}$-algebras, and twisted $\mathcal{W}$-algebras.

1.5 Exemples and series

We should now discuss of exemples of such $\mathcal{W}$-algebras. The first known exemple of $\mathcal{W}$-algebra was the $\mathcal{W}_3$ algebra, introduced in [Zam85]. It is also known to be the simplest $\mathcal{W}$-algebra, meaning the simplest non-trivial extension of the Virasoro algebra. In the notation given above, it corresponds to the algebra $\mathcal{W}(2,3)$.

This algebra is generated by two fields, the energy-momentum field T with modes (L_n) forming the Virasoro algebra, and the primary field W of conformal dimension 3 with modes (W_n).

Let's look at the commutation relations of the algebra. We necessarily have for any n, m

$$\begin{aligned} [L_m, L_n] &= (n-m)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n,-n} \\ [L_m, W_n] &= (n-2m)W_{m+n} \end{aligned} \tag{1.17}$$

Then, setting the quasi-primary field

$$\Lambda =: LL : -\frac{3}{10}\partial^2 L \tag{1.18}$$

and the structure constants

$$\begin{aligned} C^L_{WW} = 2, \quad C^\Lambda_{WW} = \frac{32}{5c+22} \\ p_{334}(m,n) = \frac{n-m}{2}, \quad p_{332}(m,n) = \frac{n-m}{60}(2m^2 - mn + 2n^2 - 8) \end{aligned} \tag{1.19}$$

We have

$$[W_m, W_n] = C^L_{WW} p_{332}(m,n)L_{m+n} + C^\Lambda_{WW} p_{334}(m,n)\Lambda_{m+n} + \frac{c}{3}\binom{n+2}{5}\delta_{n,-m} \tag{1.20}$$

We see here the appearance of a the new quasi-primary field Λ different from $: LL :$.

Since the discovery of this algebra, many other $\mathcal{W}$-algebras have been found. In particular, in the quest for the complete classification of $\mathcal{W}$-algebras, many general constructions were found, leading to large series of $\mathcal{W}$ algebras.

The first systematic construction of $\mathcal{W}$-algebras that was found after the discovery of $\mathcal{W}_3$ gives the class of $\mathcal{W}_N$ algebras, of type $\mathcal{W}(2, 3, \ldots, N)$ and of central charge $c = \frac{(3N+1)(N-1)}{2(2N+1)}$. They are general extensions of $\mathcal{W}_3$, and are constructed by each time extending the algebra $\mathcal{W}_{N-1}$ by a current of conformal dimension N.

Another general class of $\mathcal{W}$-algebras easy to find is the class of affine Kac-Moody algebras $\hat{\mathfrak{g}}_k$, which we will describe in ??. The idea is to extend the Virasoro algebra representing conformal transformations on a plane to an algebra representing conformal transformations on a semi-simple Lie algebra $\mathfrak{g}$. To each dimension of the semi-simple Lie algebra is then associated a current of conformal dimension 1, effectively extending the Virasoro algebra. The theories lying behind this class of algebras are the WZW-models.

Also considering a semisimple Lie algebra $\mathfrak{g}$, a second class of $\mathcal{W}$-algebras can be found by considering the Casimir invariants of the algebra. This lead to the so-called Casimir algebras, written $\mathcal{W}\mathfrak{g}$-algebras. It is interesting to see in particular that $\mathcal{W}\mathcal{A}_{N-1} \simeq \mathcal{W}_N$. These Casimir algebras can also be thought of as the algebra of conserved currents of a Toda field theory, or as a WZW-model where we have gauge-fixed most of the momentum of the particles, as we will see in 3.

Using this point of view, we can generalise Casimir algebras to more general forms of gauge fixing of the WZW-model using the BRST procedure, resulting in the acclaimed quantum Drinfied-Sokolov reduction. The gauge fixings depending on a sl_2 subalgebra $\mathfrak{s} \subset \mathfrak{g}$, the resulting $\mathcal{W}$-algebras were historically written $\mathcal{W}_{\mathfrak{s}}^{\mathfrak{g}}$. Understanding that $\mathfrak{s}$ only depends on two elements $(x, f) \in \mathfrak{g}$ defining a good grading, we now more often write this kind of algebra $\mathcal{W}^k(\mathfrak{g}, x, f)$, where k corresponds to the level of the reduced affine Kac-Moody algebra.

We also have ways of constructing $\mathcal{W}$-algebras from preexisting ones. First, we can use the orbifold procedure, already well-known in physics. The idea is to use an outer automorphism of a preexisting $\mathcal{W}$-algebra to enforce a symmetry on it, leading to new currents and to a bigger algebra, most often also finitely generated. For exemple, the $\mathbb{Z}_2$ orbifold of $\mathcal{W}(2,3)$ is $\mathcal{W}(2,6,8,10,12)$.

We can also take the tensor product of already known $\mathcal{W}$-algebras. For exemple we can find an algebra $\mathcal{W}(2,2,\dots 2)$ by summing n copies of the Virasoro algebra. However, the expression of the new central charge and the expression of the new energy-momentum field are not always trivial to find.

Finally, given a $\mathcal{W}$-algebra A having another $\mathcal{W}$-algebra B as subalgebra, we can consider the commutant of the subalgebra, giving rise to a new $\mathcal{W}$-algebra written A/B. This construction is called the Coset construction, and is also quite classical in CFTs. Using this construction, one can for exemple construct the serie of the D_N parafermions, whose $\mathcal{W}$-algebras are given by $\hat{so}(N)_k \oplus \hat{su}(N)_1 / \hat{so}(N)_{k+2}$.

Chapter 2

The WZW model

We should now give a gentle introduction to the WZW-model, in order to give strong natural intuition for the affine Kac-Moody algebras.

2.1 Motivation and definition

Let's first go to physics, and consider objects moving in spacetime. The relativistic dispersion relation tells us that for a general particle, its energy is given by

$$E^2 = (pc)^2 + (mc^2)^2 \tag{2.1}$$

With the usual $c = 1$, we would therefore want to consider a system where we impose

$$E^2 - p^2 - m^2 = 0 \tag{2.2}$$

Suppose we want to study the evolution of a single object. We would then want to consider its trajectory in the space of states $\mathscr{T}$, which may be given by a field $\varphi : \mathbb{R} \to \mathscr{T}$, where for $\lambda \in \mathbb{R}$ parametrizing the trajectory, $\varphi(\lambda)$ corresponds to the state of the object at λ. For exemple, in the case of a relativistic particle, we may choose $\mathscr{T} = \mathcal{M}$ to be the Minkowski space. Writing φ^t the component of φ in the time dimension and φ^a the components of φ in the space dimensions (with implicit summation), we can rewrite the above equation as

$$\begin{aligned} \left(i\partial\varphi^t\right)^2 - (-i\partial\varphi^a)^2 - (m\varphi)^2 &= 0 \\ \partial\varphi^a\partial\varphi^a - \partial\varphi^t\partial\varphi^t - m^2\varphi^2 &= 0 \\ g_{ij}\partial\varphi^i\partial\varphi^j - m^2\varphi^2 &= 0 \end{aligned} \tag{2.3}$$

with g the metric on $\mathscr{T}$, here the Lorentzian metric. We can generalize this for a relativistic particle in a curved spacetime as in general relativity, by giving g a dependance. To enforce this relation, we may choose the following action

$$\mathcal{S} = \int_{\mathbb{R}} \frac{1}{2} g_{ij}(\varphi)\partial\varphi^i\partial\varphi^j - \frac{1}{2}m^2\varphi^2 \tag{2.4}$$

Remark. We have added a $\frac{1}{2}$ factor according to the conventions, though it does not change the physics of the system.

For a massless particle, this simply reduces to

$$\mathcal{S} = \frac{1}{2}\int_{\mathbb{R}} g_{ij}(\varphi)\partial\varphi^i\partial\varphi^j \tag{2.5}$$

which corresponds to the kinetic energy of the particle.

Now, suppose the object we want to study is a string. One could simply repeat the above process, with instead $\mathscr{T} = \mathcal{M}^{\mathbb{R}}$. However, it might be more convenient to instead change the parametrization space, and consider a more general field

$$\varphi : \Sigma \to \mathscr{T} \tag{2.6}$$

where for a relativistic string we have $\Sigma = \mathbb{R}^2$ and $\mathscr{T} = \mathcal{M}$. Modifying the action accordingly, we have

$$\mathcal{S} = \frac{1}{2}\int_{\Sigma} g_{ij}(\varphi)\partial^\mu\varphi^i\partial_\mu\varphi^j \tag{2.7}$$

where we also accounted for the geometry of Σ in $\partial^\mu = \eta^{\mu\nu}\partial_\nu$, with η the metric on Σ. (2.7) is the action of the sigma model, which models arbitrary objects living in $\mathscr{T}$, parametrized by Σ.

2.2 Geometric interpretation

The action might seem complicated, but it is really just the most simple way of computing a kinetic energy for objects living on $\mathscr{T}$, accounting for the geometry of $\mathscr{T}$ and Σ. To see this, we can reformulate the model in a more geometric language. Locally, for each dimension i of $\mathscr{T}$, the $\partial_\mu\varphi^i$ are differential forms on Σ. From the point of view of Σ, we may thus write

$$\mathcal{S} = \frac{1}{2}\int_{\Sigma} \mathrm{d}\varphi \wedge *\mathrm{d}\varphi \tag{2.8}$$

But

$$\ll \alpha, \beta \gg = \int_{\Sigma} \alpha \wedge *\beta \tag{2.9}$$

is an inner product on the space of square-integrable forms on Σ. The action the reduces to

$$\mathcal{S} = \frac{1}{2} \ll \mathrm{d}\varphi, \mathrm{d}\varphi \gg \tag{2.10}$$

which makes it clear that the action is nothing more than the kinetic energy of the considered objet.

The sigma model is well defined for any two manifolds Σ and $\mathscr{T}$. Let's discuss of some classical exemples of sigma models.

For $\Sigma = \mathbb{R}^n$ and $\mathscr{T} = \mathbb{C}$, the sigma model reduces to quantum mechanics where the considered particle lives in $\mathbb{R}^n$.

For $\Sigma = \mathbb{R}$ and $\mathscr{T} = \mathbb{R}^n$, the sigma model simply describes a classical particle living in n dimensions. Switching $\mathbb{R}^n$ by $\mathscr{T} = \mathcal{M}^d$ the n-dimensional Lorentzian manifold, the sigma model instead describes a relativistic particle living in $n-1$ dimensions.

For $\Sigma = (\tau, \sigma) = \Sigma_2$ and $\mathscr{T} = \mathcal{M}$, the sigma model describes a relativistic string. More generally, for $\Sigma = \Sigma_2$ and arbitrary $\mathscr{T}$, the sigma model describes a string living on $\mathscr{T}$. This special case has been extensively studied, especially for $\mathscr{T}$ a semisimple Lie algebra. The string serves as an excitable material where the excitations can be viewed as particles, resulting in a model which can be reformulated as a free particle system.

We should make explicit the difference between the case ($\Sigma = \mathbb{R}^n, \mathscr{T} = \mathbb{C}$) and the case ($\Sigma = \mathbb{R}, \mathscr{T} = \mathbb{R}^n$). Both cases look similar, as the (almost) only difference is the switch between the parameter space and the target space. In fact, both cases describe a particle living in $\mathbb{R}^n$. However, the interpretation of the model is completely different in both cases.

The case ($\Sigma = \mathbb{R}^n, \mathscr{T} = \mathbb{C}$) is similar to what we are used to. The considered field is going from the space(time) to a space expressing some form of "quantity". The particle lives in $\mathbb{R}^n$, and the value $p \in \mathbb{C}$ of the field in $x \in \mathbb{R}^n$ indicates the probability of finding the particle at x in the space, as well as its momentum. Most of quantum field theories are based on a similar principle. For exemple, when considering the free boson discussed in the previous notes, the base space is the spacetime $\mathbb{R}^{\nvDash}$ in which the boson live, and the target space $\mathbb{R}$ can directly be interpreted as a quantity scale to indicate how much boson are in each area of the spacetime. The free fermion constitutes a similar but more complex exemple, where the value of the field is a Grassman variable. In any case, the target space constitutes a measure of probability whilst the base space constitutes the space where the particles live.

On the opposite side, when taking ($\Sigma = \mathbb{R}, \mathscr{T} = \mathbb{R}^n$), we are really looking at a particle parametrized by the base space and where the target space is the space in which the particle lives. Σ serves as a kind of proper time, which parametrizes the position of the particle. This point of view is very similar to classical mechanics, where one would usually consider the position of an object according to time:

$$\vec{x}(t) : \mathbb{R} \to \mathbb{R}^3 \tag{2.11}$$

The case ($\Sigma = \Sigma_2, \mathscr{T} = \mathcal{M}$) is also similar, describing a string living in a general manifold $\mathcal{M}$, whose excitations describe particles living in a general manifold, where the excitations in the base space serve as the measure of probability instead of the value in the target space. We wish to really emphasize the fact that usual (conformal) field theories describe a particle living in the base space, while this kind of model describes a particle living in the target space. This distinction will be very important in the future.

2.3 Sigma model on Lie algebras

We will now focus on a variant of the Sigma model defined on Lie algebras. Lie structures have been known to appear naturally in physics for a long time. Classically, in Hamiltonian mechanics, the Hamiltonian induces a sympleptic structure on the phase space, which in turn can be endowed with a Lie structure when a symmetry acts on the system. In a more general setup, symmetries are encoded by Lie structures, and act on the space of any physical system containing the symmetry. It may therefore be natural to consider the general case of strings moving on a Lie structure. Moreover, the structure of Lie algebras and in particular semisimple Lie algebra is very well known, and enable us to generate a wide variety of sigma models. For these reasons among other ones, from now on until the end of this paper, we will be considering sigma models where $\Sigma = \Sigma_2$ and where $\mathscr{T}$ is a semisimple Lie algebra.

Let $\mathfrak{g}$ be a semisimple Lie algebra, and G be its Lie group. We set $(T_a)_a$ a basis of $\mathfrak{g}$ such that

$$\mathrm{Tr}(T_a T_b) = \delta_{ab} \tag{2.12}$$

We now consider a sigma model on $\mathfrak{g}$, that is a field $\varphi : \Sigma_2 \to \mathfrak{g}$. We can decompose the field in its components φ^a, such that

$$\varphi(\tau, \sigma) = \sum_a \varphi^a(\tau, \sigma) T_a \tag{2.13}$$

The action for this field can then be written

$$\begin{aligned}\mathcal{S} &= \frac{1}{2}\int_{\Sigma_2} g_{ab}(\varphi)\partial^\mu \varphi^a \partial_\mu \varphi^b \\ &= \frac{1}{2}\int_{\Sigma_2} \partial^\mu \varphi^a \partial_\mu \varphi_a \\ &= \frac{1}{2}\int_{\Sigma_2} \mathrm{Tr}(\partial^\mu \varphi \partial_\mu \varphi)\end{aligned} \tag{2.14}$$

where we hid the geometrical informations on $\mathfrak{g}$ using the trace.

φ is a field defined on a 2 dimensional spacetime. In the future, we will want to apply 2D CFTs techniques to it. In preparation for this, we may already take the usual coordinates used in 2D CFTs. As is done conventionally, we define

$$\begin{aligned} z &\equiv \tau + i\sigma \qquad & \partial &\equiv \frac{1}{2}(\partial_\tau - i\partial_\sigma) \\ \bar{z} &\equiv \tau - i\sigma \qquad & \bar{\partial} &\equiv \frac{1}{2}(\partial_\tau + i\partial_\sigma)\end{aligned} \tag{2.15}$$

Assuming Σ_2 is flat[1], the metric becomes

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \tag{2.16}$$

In term of these coordinates, the action can be written as

$$\mathcal{S} = 2\int_{\Sigma_2} \mathrm{Tr}(\partial\varphi\bar{\partial}\varphi) \tag{2.17}$$

Let's now try to reformulate this model on the Lie group instead of the Lie algebra. We first need to define g the pullback of φ onto G:

$$\begin{aligned} & g : \Sigma_2 \to G \\ & g(z, \bar{z}) = e^{\varphi(z,\bar{z})} = e^{\varphi^a(z,\bar{z}) T_a}\end{aligned} \tag{2.18}$$

[1]the curvature of Σ_2, if taken as a dynamic variable, makes a Gauge redundancy. Taking a flat metric for Σ_2 is thus a kind of gauge fixing

Noticing that

$$\partial_\mu g = (\partial_\mu \varphi(z,\bar{z}))\, g = g\partial_\mu \varphi(z,\bar{z}) \tag{2.19}$$

we may rewrite the action as

$$\mathcal{S} = 2\int_{\Sigma_2} \mathrm{Tr}(g^{-1}\partial g\; g^{-1}\bar{\partial} g) \tag{2.20}$$

This form of the action is easily understood geometrically, knowing that $g^{-1}\mathrm{d}g$ is the Maurer-Cartan form carrying all of the geometry of $\mathfrak{g}$, and recalling that the Lagrangian density of S is nothing but the pullback on Σ_2 of the metric of $\mathfrak{g}$. With the action in this form, we understand that our model can also be seen as a string moving on the group manifold G.

From now on, we will add a coupling parameter to the action. We will also compactify Σ_2 into $\mathbb{S}^2$, as is usually done in conformal field theory. Note that both of these operations do not affect the dynamics of the system. We set

$$\mathcal{S} = \frac{1}{\lambda^2}\int_{\mathbb{S}^2} \mathrm{Tr}(g^{-1}\partial g\; g^{-1}\bar{\partial} g) \tag{2.21}$$

Or equivalently,

$$\mathcal{S} = \frac{1}{4\lambda^2}\int_{\mathbb{S}^2} \mathrm{Tr}(g^{-1}\partial^\mu g\; g^{-1}\partial_\mu g) \tag{2.22}$$

This corresponds exactly to the action of the so called principal chiral model. We will now see the explanation behind this name.

2.4 The $G \times G$ symmetry

In view of (2.21), a $G \times G$ global symmetry also appear almost explicitely. Indeed, let $(g_L, g_R) \in G \times G$. Then under the global transformation

$$g \to g_L g g_R \tag{2.23}$$

$$\begin{aligned} \mathcal{S}' &= \frac{1}{\lambda^2}\int_{\mathbb{S}^2} \mathrm{Tr}(g_R^{-1} g^{-1} g_L^{-1}\partial(g_L g g_R)\; g_R^{-1} g^{-1} g_L^{-1}\bar{\partial}(g_L g g_R)) \\ &= \frac{1}{\lambda^2}\int_{\mathbb{S}^2} \mathrm{Tr}(g_R^{-1} g^{-1} g_L^{-1} g_L \partial g\; g_R g_R^{-1} g^{-1} g_L^{-1} g_L \bar{\partial} g\; g_R) \\ &= \mathcal{S} \end{aligned} \tag{2.24}$$

The actions remains unchanged. This symmetry was expected. In a flat target space ($\mathscr{T} = \mathbb{R}^n$), this corresponds to the usual invariance by translation. Here, the $G \times G$ symmetry simply is a Lie group equivalent of the translational invariance, saying that the action doesn't change if we move our system using the left or right action of G on itself.

Remark. G acts independently on the right and on the left of the field, so this kind of symmetry is called a chiral symmetry. Moreover, the considered string lives on G as a manifold. The string doesn't see the Lie group structure on G, only its action on itself. As such, this model describes a massless string living on a principal homogeneous space of G, a system on which there is a chiral G symmetry. This is the reason behind the name "principal chiral model".

Let's look at the conserved current associated to this symmetry. Classically, the conserved current associated to spatial translations is the momentum, and the conserved current associated to a time translation is energy. We therefore expect to get a current which we can interpret similarly to energy-momentum.

We consider a variation of the field $g \to g + \delta g$. Under this, the quantities of study transform as follow

$$\begin{aligned} g^{-1} &\to g^{-1} - g^{-1}\delta g g^{-1} \\ \partial_\mu g &\to \partial_\mu g + \delta \partial_\mu g \end{aligned} \tag{2.25}$$

With a few computations, we get

$$\begin{aligned} \delta\mathcal{S} &= \frac{1}{2\lambda^2}\int_{\mathbb{S}^2} \mathrm{Tr}\left((-g^{-1}\delta g g^{-1}\partial_\mu g + g^{-1}\delta\partial_\mu g)g^{-1}\partial^\mu g\right) \\ &= -\frac{1}{2\lambda^2}\int_{\mathbb{S}^2} \mathrm{Tr}\left(g^{-1}\delta g \partial_\mu(g^{-1}\partial^\mu g)\right) \end{aligned} \tag{2.26}$$

Which gives as a byproduct the equation of motion

$$\partial_\mu(g^{-1}\partial^\mu g) = 0 \tag{2.27}$$

Let's now specialize the transformation to the case of a right action of G. Let $\varepsilon_g = e^{\epsilon^a T_a}$ be an infinitesimal element of G, and let us consider the transformation

$$g \to g\varepsilon_g = g e^{\epsilon^a T_a} \sim g + g\epsilon^a T_a \tag{2.28}$$

We may identify $\delta g = g\epsilon^a T_a$ in (2.26). We see that the resulting conservation equation is nothing more than the equation of motion, and therefore the conserved current associated to the right action of G is

$$J^\mu = g^{-1}\partial^\mu g \tag{2.29}$$

We can proceed similarly for the left action. We have $\delta g = \epsilon^a T_a g$ which, insterted in (2.26) and using the cyclicity of the trace, results in the following conserved current

$$\tilde{J}^\mu = \partial^\mu g\, g^{-1} \tag{2.30}$$

We see that J^μ corresponds to the Maurer-Cartan form of G, and $\tilde{J}^\mu$ is also very similar, both giving information on the length of the string taking into account the geometry of G. We can therefore interpret both of these currents as momentums of the string, as was expected from the classical case.

2.5 Conformal invariance and geometric obstructions

Let's imagine our string moving on a manifold. The string is considered free and massless, as the field of a free boson. The freedom of the string makes it invariant under spatial translations on G as saw earlier, which leads to the $G \times G$ symmetry. But this freedom should also make it more or less rotational invariant[2]. Moreover, the string is massless, making it supposedly invariant

[2] We are nuancing this invariance with "more or less", as this invariance is obviously very dependent on the geometry of G and is not as straightforward as in the classical case.

under rescaling. These assumed invariance lead us to think that the string should act invariant under conformal transformations on G.

Note that this form of conformal invariance is very different from usual conformal invariance, as the conformal transformations under which the system is invariant do not act on the same space. In the theory of the free boson for exemple, the boson is invariant under conformal transformation of the space it lives on, the base space. Here, the string lives on the target space and should therefore be invariant under some kind of conformal invariance of the target space.

How does can this invariance manifest itself? Recall that we are considering this string as a worldline, host for excitations and therefore particles. At the end of the day, we are considering a quantum field coming from a 2 dimensional base space, that is a 2D quantum field theory. Particles, or excitations, are able to move on the string (itself moving on G) going right or left. But due to the masslessness of the string and more generally due to conformal invariance, the particles should always move at a fixed speed, and should not be able to turn back. As such, there should be two decoupling momentums, one accounting for the excitations of the string going left and one accounting for those going right.

This is a well known phenomenom in 2D CFTs. The left and right moving dimensions correspond to z and $\bar{z}$, the holomorphic and antiholomorphic dimensions. The energy momentum of a field in a 2D CFT is given by the energy-momentum tensor $T_{z\bar{z}}$, which answers to the following equations of motion

$$\begin{aligned} \partial T_{\bar{z}z} + \bar{\partial} T_{zz} &= 0 \\ \partial T_{\bar{z}\bar{z}} + \bar{\partial} T_{z\bar{z}} &= 0 \end{aligned} \tag{2.31}$$

But the conformal symmetry of the system makes the energy-momentum traceless, reducing the above equations to

$$\bar{\partial} T_{zz} = 0 \qquad \partial T_{\bar{z}\bar{z}} = 0 \tag{2.32}$$

Which leads to the decoupling of the holomorphic quantity $T = T_{zz}$ and the antiholomorphic quantity $\bar{T} = T_{\bar{z}\bar{z}}$, representing the momentums of the left and right moving particles.

In our case, the tensors accounting for the momentum of the string on the manifold, or accounting for the momentum of the excitations (particles) of the string on the manifold, are J^μ and $\tilde{J}^\mu$. Focusing on one current, recalling (2.16), we have the conservation equation

$$\partial J_{\bar{z}} + \bar{\partial} J_z = 0 \tag{2.33}$$

With respect to the discussion we just had on 2D CFTs, we should expect both derivatives to vanish separately.

However, they do not vanish separately, showing that our system is in fact not conformally invariant at the quantum level. Indeed, G is a connected Lie group so we can make J^μ diseappear through the action of G. But the field strength of J^μ is invariant through the action of symmetries, and so we have

$$\partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu] = 0 \tag{2.34}$$

If both $\partial J_{\bar{z}}$ and $\bar{\partial} J_z$ did vanish, then the dual current $\epsilon_{\mu\nu} J^\nu$ would also be conserved. This would lead to

$$\begin{aligned} 0 &= \partial_\mu(\epsilon^{\mu\nu} J_\nu) \\ &= \frac{1}{2}\epsilon^{\mu\nu}(\partial_\mu J_\nu - \partial_\nu J_\mu) \\ &= -\frac{1}{2}\epsilon^{\mu\nu}[J_\mu, J_\nu] \\ &= [J_\nu, J_\mu] \end{aligned} \tag{2.35}$$

This is true if and only if the Lie algebra is abelian, meaning if it is completely flat. We already knew that this system is conformally invariant when the target space is flat, by analogy with usual free particle systems. What (2.35) tells us is that the geometry of G can locally constitute an obstruction to the conformal invariance of the system. It is in fact not too hard to imagine how the topological artefacts of a general manifold can obstruct rotational invariance or rescaling invariance.

2.6 The Wess-Zumino term

Upon facing this issue, we would want to upgrade our model to remove this obstruction. Witten found a way to solve this issue by adding the celebrated Wess-Zumino term, leading to the Wess-Zumino-Witten model, or WZW model.

The first step into solving this obstruction is to extend the field g. This field is defined on $\mathbb{S}^2$, which can be seen as the boundary of the ball B in 3 dimensions. We know that maps from $\mathbb{S}^2$ to G are defined, up to homotopy, by the 2^{nd} homotopy group $\pi_2(G)$, which is trivial for any Lie group. As such, g is homotopic to a constant map, which can be extended to a map from B to G. It is therefore clear that we can extend g into a map $g : B \to G$.

Let's now think about what kind of term could be defined using this extension. Up until now, the action simply represented the kinetic energy of the string, which can also be seen as its volume in G. As a matter of fact, the Lagrangian was nothing more than a pullback of the metric of G onto $\mathbb{S}^2$, using the Maurer-Cartan form $g^{-1}\mathrm{d}g$. This form brings a vector $v \in T_x G$ at x onto $T_e G = \mathfrak{g}$, thus transporting informations about the metric in x to $\mathfrak{g}$. If we want to generalize this to the 3-dimensional ball, we could therefore consider the 3-form $\wedge^3 g^{-1}\mathrm{d}g$ which brings a 3-vector at x onto $\mathfrak{g}$. Using this, one can pullback the volume form with a term looking like

$$\mathrm{Tr}(\wedge^3 g^{-1}\mathrm{d}g) \tag{2.36}$$

It may thus be natural to consider the following term

$$\begin{aligned} \tilde{\mathcal{S}} &= \int_B \mathrm{Tr}(\wedge^3 g^{-1}\mathrm{d}g) \\ &= \frac{1}{6}\int_B \epsilon_{\alpha\beta\gamma}\mathrm{Tr}(g^{-1}\partial^\alpha g\; g^{-1}\partial^\beta g\; g^{-1}\partial^\gamma g) \end{aligned} \tag{2.37}$$

which is nothing more than a kinetic term generalized to the ball, to maybe bypass the geometric obstruction.

However, we must remember that the extension of g onto B is not unique. Let's see if the term $\tilde{\mathcal{S}}$ can define a proper theory. Classically, the equations of motion are found by varying the field g by an infinitesimal transformation $g \to g+\delta g$. Let's compute the variation of $\tilde{\mathcal{S}}$ for such a transformation, similarly to what we have done in (2.26) and using Stokes' theorem:

$$\begin{aligned} \delta\tilde{\mathcal{S}} &= 3\int_B \mathrm{Tr}((-g^{-1}\delta g g^{-1}\mathrm{d}g + g^{-1}\mathrm{d}\delta g) \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g) \\ &= 3\int_B \mathrm{d}\mathrm{Tr}(g^{-1}\delta g g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g) \\ &= 3\int_{\mathbb{S}^2} \mathrm{Tr}(g^{-1}\delta g g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g) \\ &= 3\int_{\mathbb{S}^2} \mathrm{Tr}(g^{-1}\delta g \mathrm{d}(g^{-1}\mathrm{d}g)) \end{aligned} \tag{2.38}$$

As such, if g is fixed on $\mathbb{S}^2$ meaning $\delta g|_{\mathbb{S}^2} = 0$, then $\delta\tilde{\mathcal{S}}$ vanishes. This shows that the equations of motions are well defined for this term, at least at a classical level. A classical model could therefore be defined by simply adding $\lambda\tilde{\mathcal{S}}$ to the action, without any modification, for any λ acting as a coupling constant.

However, it might not always be the case at the quantum level. At the quantum level, we must also consider the case where two extensions of g are not homotopic, as the field may jump from one topological state to another. Let 2 extensions of g on B. We can glue them together on the boundary, as g is known on $\mathbb{S}^2$. This gives a map

$$g : (B \sqcup B)/\partial B \approx \mathbb{S}^3 \to G \tag{2.39}$$

Such map are classified up to homotopy by $\pi_3(G)$, which is isomorphic to $\mathbb{Z}$ for any simple compact Lie group. Even stronger, any map $g : \mathbb{S}^3 \to G$ is homotopic to a map $g : \mathbb{S}^3 \to SU(2) \approx \mathbb{S}^3$ if G has a $SU(2)$ subgroup. The index in $\mathbb{Z}$ given by the homotopy group then simply tells how many times $\mathbb{S}^3$ wraps around itself.

We have seen that $\tilde{\mathcal{S}}$ is invariant under homotopies, and that the homotopic classes of extensions are classified by an integer. Knowing that, we would like to compute $\Delta\tilde{\mathcal{S}}$ the difference of $\tilde{\mathcal{S}}$ for two neighbooring extensions. To compute it, we may take $g : \mathbb{S}^3 \subset \mathbb{R}^4 \to SU(2)$ as follow

$$\begin{aligned} g(y) &= y^0 - iy^k\sigma_k \\ g^{-1}\partial^k g &= -i\sigma^k \end{aligned} \tag{2.40}$$

Then $\mathbb{S}^3$ wraps around itself exactly once, so we can compute

$$\begin{aligned} \Delta\tilde{\mathcal{S}} &= \frac{1}{6}\int_B \epsilon_{\alpha\beta\gamma}\mathrm{Tr}(g^{-1}\partial^\alpha g\ g^{-1}\partial^\beta g\ g^{-1}\partial^\gamma g) \\ &= \frac{1}{6}(-i)^3 2\pi^2 \sum_{i,j,k} \epsilon_{ijk}\mathrm{Tr}(\sigma^i\sigma^j\sigma^k) \\ &= -(2\pi)^2 \end{aligned} \tag{2.41}$$

The computations were done fastly, but a more detailed computation can be found in [Ebe19]. Now in quantum field theories, we use the action to compute correlators, which for a string of operators $\mathcal{O}$ take the form[3]

$$\langle \mathcal{O} \rangle = \int [\mathrm{d}g] \mathcal{O}(g) e^{i\mathcal{S}(g)} \tag{2.42}$$

This is well defined if the exponential under the integral is single-valued, so if $\mathcal{S}$ is single-valued modulo 2π. If we want to add a topological term as $a\tilde{\mathcal{S}}$ to our action with a a constant, we must add it in such a way that $a\tilde{\mathcal{S}}$ is single-valued modulo 2π, which imposes

$$\begin{aligned} a\Delta\tilde{\mathcal{S}} &= 2k\pi \\ a &= -\frac{k}{2\pi} \end{aligned} \tag{2.43}$$

for some integer $k \in \mathbb{Z}$. This brings us to considering the following new, modified action

$$\mathcal{S}' = \mathcal{S} - \frac{k}{2\pi}\tilde{\mathcal{S}} \tag{2.44}$$

which is well defined at the classical and quantum level for integer k. This new action is composed of the kinetic term from the Sigma model, to which we have added a topological term in hope it might help us decouple the holomorphic and antiholomorphic parts of the currents J^μ and $\tilde{J}^\mu$.

2.7 The WZW model

Now that we have modified the action and proved it is well defined, let's see if it actually helps the situation. Let's first see if this action is still invariant under the global $G \times G$ action. Let $(g_L, g_R) \in G \times G$. Under the global transformation

$$g \to g_L g g_R \tag{2.45}$$

Recalling that $\mathcal{S}$ is invariant under this transformation, $\mathcal{S}'$ becomes

$$\begin{aligned} \mathcal{S}' \to \mathcal{S} - \frac{k}{12\pi} \int_B & \epsilon_{\alpha\beta\gamma} \mathrm{Tr}(g_R^{-1} g^{-1} g_L^{-1} g_L \partial^\alpha g \\ & g_R g_R^{-1} g^{-1} g_L^{-1} g_L \partial^\beta g \; g_R g_R^{-1} g^{-1} g_L^{-1} g_L \partial^\gamma g \; g_R) \\ \to \mathcal{S} - \frac{k}{2\pi}\tilde{\mathcal{S}} = \mathcal{S}' & \end{aligned} \tag{2.46}$$

So the system still contains a $G \times G$ chiral symmetry. We can then apply the same techniques as we did previously for the sigma model. Reading the current from (2.26) and (2.38) and adding the right coefficient in front of each term, we see that the current associated to the right action of G is still the same, but its conservation equation has become:

$$\left(1 + \frac{\lambda^2 k}{\pi}\right) \partial(g^{-1}\bar{\partial}g) + \left(1 - \frac{\lambda^2 k}{\pi}\right) \bar{\partial}(g^{-1}\partial g) = 0 \tag{2.47}$$

[3] Actually, this formulation doesn't make a consensus. Some people take the exponential to be $e^{-\mathcal{S}(g)}$, and other choose $e^{-i\mathcal{S}(g)}$. It is only a matter of convention for the factor in front of the action, but it will affect the expression of our algebra later on. It is therefore important to remember that the expression of the action of the WZW-model can vary from one author to the other. Here, we stay consistent with the previous lecture notes.

and similarly for the left action of G. We now see that compared to (2.33), there are coefficients in front of the currents, which can be annihilated by a wise choice of λ for $k \neq 0$. Without loss of generality, even if it means exchanging the holomorphic and antiholomorphic dimensions, we can suppose $k > 0$. Then, we will choose $\lambda^2 = \frac{\pi}{k}$, such that we have

$$\partial(g^{-1}\bar{\partial}g) = 0 \tag{2.48}$$

thus defining the quantity $\bar{J} = \bar{J}(\bar{z}) = kg^{-1}\bar{\partial}g$ which only depends on the antiholomorphic dimension. Similarly, from the left action of G, we obtain with $k > 0, \lambda = \frac{\pi}{k}$

$$\bar{\partial}(\partial g g^{-1}) = 0 \tag{2.49}$$

which gives the holomorphic quantity $J = J(z) = k\partial g g^{-1}$.

Remark. Just as in a usual 2D CFT where one find the holomorphic current T and holomorphic current $\bar{T}$ due to the symmetry through translation, we find two currents through the symmetry given by the G action, which corresponds in a Lie group to translations. However, we should note the dissymmetry between the definition of J and $\bar{J}$.

The model thus defined is the acclaimed WZW model of level k on the Lie group G. Its action reads[4]

$$\mathcal{S}_{\text{WZW}} = \frac{k}{\pi}\int_{\mathbb{S}^2} \text{Tr}(g^{-1}\partial g\, g^{-1}\bar{\partial}g) - \frac{k}{2\pi}\int_B \text{Tr}(\wedge^3 g^{-1}\text{d}g) \tag{2.50}$$

We have seen how the Wess-Zumino term enhances the principal chiral model by decoupling the holomorphic and antiholomorphic currents. But how can this term be interpreted? What is its nature? To answer these questions, we need to see that $\text{Tr}(\wedge^3 g^{-1}\text{d}g)$ is a 3-form, and therefore a closed form, on B. Knowing furthermore that B is simply connex, we can find a 2-form X such that

$$\text{d}X = \text{Tr}(\wedge^3 g^{-1}\text{d}g) \tag{2.51}$$

Resulting in

$$\begin{aligned}\int_B \text{Tr}(\wedge^3 g^{-1}\text{d}g) &= \int_B \text{d}X \\ &= \int_{\mathbb{S}^2} X\end{aligned} \tag{2.52}$$

We may then find $X(g)^{\mu\nu}$ such that we can rewrite this term as

$$\int_{\mathbb{S}^2} \epsilon_{\alpha\beta} X(g)^{\mu\nu}\partial^\alpha g_\mu \partial^\beta g_\nu \tag{2.53}$$

From this perspective, we see that the Wess-Zumino term is just an antisymmetric coupling of the string.

In this model, the strictly positive integer k is called the *level* of the model. It appears in front of the kinetic term and in front of the topological term. In front of the kinetic term, it may be simply seen as an external adjustable

[4]Due to the difference in convention for the expression of the path-integral, there is also a difference in convention here for the factor in front of the Wess-Zumino term.

constant, that must be fixed this way to perfectly compensate for the other term and to ensure conformal invariance on G. However, it has more meaning than a simple coupling constant in front of the topological term. We have seen that k must be an integer to ensure the exponential is single-valued. k can then be seen as answering to a kind of monodromy condition, which defines the considered sector in the theory, just like in the case of winding sectors for the free boson on the torus. Here, instead of having a winding number counting the number of turns made around the torus, k can be abstractly seen as a winding number counting the "number of turns" made around the geometrical obstructions of the Lie group.

2.8 Extended symmetry

Let G a simple Lie group, k a strictly positive integer. We consider the WZW model of level k defined on G. Most of the discussion we will have can be generalized easily to semisimple Lie algebra. However, we will only need simple Lie algebras later on. We have shown that this model is invariant under the right and left global action of G, symmetry inherited from the Sigma model. In fact, adding the topological term even enhanced this symmetry. Consider a small perturbation of g:

$$g \to g + \delta g \tag{2.54}$$

Then proceeding as we did to get (2.47), we get the following perturbation of $\mathcal{S}$:

$$\delta\mathcal{S} = 2\int_{\mathbb{S}^2} \mathrm{Tr}\left(g^{-1}\delta g \partial(g^{-1}\bar{\partial}g)\right) = 0 \tag{2.55}$$

The main difference with what we had previously (2.26) is that this equation only looks at the derivative of $\bar{J}$ along the holomorphic dimension. Therefore, if we specialize this transformation to a right action of G local with respect to the antiholomorphic dimension by $\varepsilon_g(\bar{z}) = e^{\epsilon^a(\bar{z})T_a}$, we have

$$\begin{aligned}\delta\mathcal{S} &= 2\int_{\mathbb{S}^2} \mathrm{Tr}\left(g^{-1}g\epsilon^a(\bar{z})T_a\partial(g^{-1}\bar{\partial}g)\right) = 0\\ &= 2\int_{\mathbb{S}^2} \mathrm{Tr}\left(\partial(\epsilon^a(\bar{z})T_a g^{-1}\bar{\partial}g)\right)\\ &= 0\end{aligned} \tag{2.56}$$

So $\mathcal{S}$ is invariant under this kind of transformation. Similarly, $\mathcal{S}$ is invariant under the left action of G local with respect to the holomorphic dimension. Therefore, the $G\times G$ symmetry of the system has been upgraded to a $G(z)\times G(\bar{z})$ symmetry.

Remark. Note how this allows us to easily find the solutions of the WZW-model. Indeed, for any solution $g_0(z,\bar{z})$ of the model and for any chiral functions $g_L(z)$ and $g_R(\bar{z})$ with values in G, the $G(z)\times G(\bar{z})$ symmetry forces the modified field $g_1 = g_L g_0 g_R$ to still be a solution of the system. Noticing that the trivial field

$$g_0 : (z,\bar{z}) \to 1 \tag{2.57}$$

is a solution of the system, we see that any field of the form

$$g(z,\bar{z}) = g_L(z)g_R(\bar{z}) \tag{2.58}$$

is also a solution of the system. It is in fact not too hard to show that any solution of the theory takes this form.

Considering the chiral $G(z) \times G(\bar{z})$ symmetry of the theory, we may see similarities with usual 2D conformal field theories, but where here the "conformal invariance" is on the target space instead of the base space.

In our theory, the $G \times G$ symmetry corresponding to translations on the Lie algebra got upgraded to a $G(z) \times G(\bar{z})$ symmetry where in each chiral dimension, the action acts as an analytically local translation, that is as the left or right action of G depending analytically on the dimension. Similarly in a 2D CFT, the conformal group action decouples in the 2 chiral dimensions, by acting on each dimension as a general analytical function. This action can also be seen as a generalisation of the global translation symmetry on the base space, where instead of a global transformation

$$\varphi(z) \to \varphi(z + z_0) \tag{2.59}$$

The symmetry acts as local analytical translations of the form

$$\varphi(z) \to \varphi(f(z)) \tag{2.60}$$

We can also view this infinite dimensional freedom as a consequence of the decoupling of the chiral energy-momentum currents, as follow. Let's first consider the case of a 2D CFT, and let's look at its holomorphic dimension. The decoupling of the energy-momentum is written

$$\bar{\partial} T = 0 \tag{2.61}$$

Therefore, we can define the classically conserved charge

$$Q = \frac{1}{2\pi i} \int_{\mathcal{C}_z} \mathrm{d}z \; T(z) \tag{2.62}$$

Because T depends on z, we can expand it in modes as

$$T(z) = \sum_n L_n z^{-n-2} \tag{2.63}$$

We recognize the original conserved charge $Q = L_{-1}$, associated to translations. But recalling (2.61) we see that due to the fact that (2.61) only depends on the antiholomorphic dimension, all $(L_n)_n$ are conserved in the holomorphic dimension, leading to the chiral representation of the Virasoro algebra. This way, we see that the existence of the two chiral algebras $\mathrm{Vir} \times \overline{\mathrm{Vir}}$ classically representing the conformal group is due to the decoupling of the two currents. However, it is important to note that the conformal symmetry leads only to the presence of this Virasoro algebra. It does not mean this algebra is a symmetry of the model. Especially, only L_0 generates a symmetry of the system as it is the only one to commute with the Hamiltonian $L_0 + \bar{L}_0$.

The same discussion can be made for our WZW-model. We can focus on the holomorphic dimension, knowing the same things happen in the antiholomorphic dimension. The holomorphic conserved current is $J(z)$, which can be decomposed in multiple conserved currents on a basis of $\mathfrak{g}$:

$$J(z) = J^a(z) T_a \tag{2.64}$$

Using this decomposition, the decoupled equation of motion rewrites

$$\bar{\partial} J^a(z) = 0 \tag{2.65}$$

But then, expanding each current into its modes

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1} \tag{2.66}$$

we see that all of the $(J^a_n)_{n,a}$ are conserved charges in the holomorphic dimension, associated to the extended symmetry given by the left action of $G(z)$. Similarly, we have the charges conserved in the antiholomorphic dimension $(\bar{J}^a_n)_{n,a}$, associated to the extended symmetry given by the right action of $G(\bar{z})$. Note once again that they do not generate the symmetry, but are only hints of it.

2.9 The affine Kac-Moody algebra

As we saw in the previous subsection, due to the decoupling of the right and left moving momentums, the family of operators $(J^a_n)_{n,a}$ is always present in the operator space, much like the space of operators always contains a representation of the Virasoro algebra in a 2D CFT. Understanding this family then leads to a good comprehension of the Hilbert space of the theory, as we will discuss in 2.11. This motivates why we will now study the algebra lying behind the currents and their modes.

The WZW model is a 2D QFT, so we may freely use known results on such theories to study the relations between the currents. Let us begin with the OPE of the currents $J^a(z)$. These currents obviously have antiholomorphic dimension 0. But they are conserved currents, so their conformal weights is $(1,0)$. We may assume that the identity is the only field of conformal weights $(0,0)$. The original symmetry forms a closed algebra, so their OPE should also be closed. Moreover, we can suppose the symmetry is unitary, such that the OPE should not contain operators with negative conformal dimension. Knowing these, we may write in a very general fashion

$$J^a(z)J^b(w) \sim \frac{\kappa^{ab}}{(z-w)^2} + \frac{h^{ab}_{\ \ c} J_c(w)}{z-w} \tag{2.67}$$

But due to associativity of the OPE and symmetry under the exchange of J^a and J^b, we get

$$\begin{gathered} \kappa^{ab} = \kappa^{ba} \qquad h^{ab}_{\ \ c} = -h^{ba}_{\ \ c} \\ \kappa^{cd} h^{ab}_{\ \ d} = \kappa^{bd} h^{ca}_{\ \ d} = \kappa^{ad} h^{bc}_{\ \ d} \\ h^{ab}_{\ \ d} h^{dc}_{\ \ e} + h^{bc}_{\ \ d} h^{da}_{\ \ e} + h^{ca}_{\ \ d} h^{db}_{\ \ e} = 0 \end{gathered} \tag{2.68}$$

This implies that the $h^{ab}_{\ \ c}$ are actually the structure constants of a Lie algebra, and that κ^{ab} is a symmetric invariant tensor on this Lie algebra. In view of our model, the Lie algebra in question must be $\mathfrak{g}$. Moreover, the only symmetric invariant tensor on simple Lie algebra is, up to a scale, the Killing form (which is just the trace). But we have chosen a basis on $\mathfrak{g}$ such that the trace acts as the identity on them. Therefore, we have

$$\kappa^{ab} = c\delta^{ab} \tag{2.69}$$

for some constant c. It turns out c is actually the level of the model k. With $f^{ab}_{\ c}$ the structure constants of $\mathfrak{g}$, we can synthetize this discussion by writing the final form of the OPE

$$J^a(z)J^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \frac{f^{ab}_{\ c} J_c(w)}{z-w} \tag{2.70}$$

This OPE defines an algebra structure on the currents, called the *current algebra.*

Doing the same operations on the antiholomorphic currents, we see that our theory actually contains 2 copies of the current algebra.

$$\bar{J}^a(\bar{z})\bar{J}^b(\bar{w}) \sim \frac{k\delta^{ab}}{(\bar{z}-\bar{w})^2} + \frac{f^{ab}_{\ c} \bar{J}_c(\bar{w})}{\bar{z}-\bar{w}} \tag{2.71}$$

Know that we know the structure of the current algebras, we can compute the structure of the modes. We expand each current in its modes as

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1} \tag{2.72}$$

Remark. Note that we shift the index of J^a_n by one in concordance with conventions, due to the fact that J^a is a current of conformal dimension 1.

By means of contour integral, we can then compute the commutation relations of the J^a_n, as is done classically in the radial quantization. We have

$$\begin{aligned}[J^a_m, J^b_n] &= \frac{1}{(2\pi i)^2}\left(\oint \mathrm{d}z \oint_{|z|>|w|} \mathrm{d}w - \oint \mathrm{d}z \oint_{|z|<|w|} \mathrm{d}w\right) z^m w^n J^a(z)J^b(w) \\ &= km\delta^{ab}\delta_{m+n,0} + f^{ab}_{\ c} J^c_{m+n}\end{aligned} \tag{2.73}$$

These commutation relations define a non-linear affine Lie algebra, called the *affine Kac-Moody algebra.* In a general fashion, one can define the affine Kac-Moody algebra of level k of any semisimple Lie algebra $\mathfrak{g}$ by following this procedure. We usually write it $\hat{\mathfrak{g}}_k$. We should not forget that these relations define the commutation relations of the conserved charges of our system, associated to the local translational symmetries. The non-linear term proportional to k corresponds to the topological modification we have forced on the system, as to bypass the obstructions to the conformal invariance on G made by the geometry of G. These relations may therefore be seen as a "projection" of the geometry of G onto the space of group manifolds where the theory of a free string is conformally invariant. It is a form of regularization of the geometry of G.

2.10 The Sugawara construction

So far, we have shown that adding the Wess-Zumino term to the Sigma model made the string and its excitations "conformally invariant"[5], meaning that we

[5] I quote "conformally invariant" because although the symmetry of the WZW-model looks like a conformal group symmetry, it is not the actual conformal group acting on the system but its equivalent on Lie groups.

may reparametrize the space on which lives the string using conformal transformations, without changing the dynamics of the string. Switching our point of view, this should also imply that we can reparametrize the string without changing the dynamics of the system, meaning the conformal group[6] should act on the parameter space as a symmetry. But in our theory, the parameter space is the base space, and this just mean our theory should also be conformally invariant in the usual sense. In other words, the $G(z) \times G(\bar{z})$ symmetry should imply that the system is a 2D CFT, and the presence of the chiral affine Kac-Moody algebras in the theory should imply the presence of the chiral Virasoro algebras as well.

Let's prove that the Virasoro algebra is actually represented in the WZW-model. To do so, we should find the energy-momentum tensor of the theory, associated to translations on the string and along time instead of translations on G. Classically, the energy of a free particle is given by half the square of its momentum, such that we may classically want

$$\tilde{T}(z) = \frac{1}{2k} J^a(z) J_a(z) \tag{2.74}$$

However, expert as we are of quantum theories, we know better than this and we know that we should take the normal ordering of this, as to avoid unphysical states. We will therefore consider

$$\tilde{T}'(z) = \frac{1}{2k} : J^a J_a : (z) \tag{2.75}$$

Notice how this is very similar to the theory of the free boson, where $T(z) \propto : \partial\varphi(z)\partial\varphi(z) :$. Knowing that our theory is supposed to describe free particles living on G, this is a good sign. We may however not be sure of the constant in front of $: J^a J_a : (z)$. To fix it, let's compute the OPE of $\tilde{T}'$ with the currents J^a. We are only interested in the singular terms, and will use this fact in our computations. We will denote the singular part of the OPE by a contraction. We have

$$\begin{aligned} \overline{J^a(z)\tilde{T}'}(w) &= \frac{1}{2k}\frac{1}{2\pi i}\oint_w \frac{\mathrm{d}x}{x-w} \overline{J^a(z)(J^b(x)}J^b(w)) \\ &= \frac{1}{2k}\frac{1}{2\pi i}\oint_w \frac{\mathrm{d}x}{x-w} (\overline{J^a(z)J^b}(x)J^b(w) + J^b(x)\overline{J^a(z)J^b}(w)) \end{aligned} \tag{2.76}$$

Inserting the OPE given in (2.70), we get

$$\begin{aligned} \overline{J^a(z)\tilde{T}'}(w) &= \frac{1}{2k}\frac{1}{2\pi i}\oint_w \frac{\mathrm{d}x}{x-w}\left[\frac{k\delta^{ab}}{(z-x)^2} + \frac{f^{ab}_c J^c(x)}{z-x}\right] J^b(w) \\ &+ \frac{1}{2k}\frac{1}{2\pi i}\oint_w \frac{\mathrm{d}x}{x-w} J^b(x)\left[\frac{k\delta^{ab}}{(z-w)^2} + \frac{f^{ab}_c J^c(w)}{z-w}\right] \end{aligned} \tag{2.77}$$

The second order poles don't contribute to the final expression since the kronecker delta are symmetric and the structure constant completely antisymmetric. Expanding further, we can finally compute

$$\overline{J^a(z)\tilde{T}'}(w) = \frac{1}{2k}\left(\frac{2k\delta^{ab}J^b(w)}{(w-z)^2} + \frac{f^{ab}_c f^{cb}_d J^d(w)}{(w-z)^2}\right) \tag{2.78}$$

[6]This time the actual one

$f_c^{ab} f_d^{cb}$ is a 2-invariant tensor, so it is proportional to the Killing form which is in our basis the identity. This defines the Coxeter number $h^\vee$:

$$f_c^{ab} f_d^{cb} \equiv h^\vee \delta^{ad} \tag{2.79}$$

With this definition, we have

$$\begin{aligned} \tilde{T}'(z) J^a(w) &= J^a(w) \tilde{T}'(z) \\ &= \frac{k + h^\vee}{k} \frac{J^a(z)}{(w-z)^2} \\ &= \frac{k + h^\vee}{k} \left(\frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} \right) \end{aligned} \tag{2.80}$$

This is very close to what we would like to have for the OPE between our current and an energy momentum-tensor. The only wrong part is the factor in front, which should be 1 since currents have conformal dimension 1. To correct this, we finally define

$$T(z) = \frac{1}{2(k + h^\vee)} : J^a J_a : (z) \tag{2.81}$$

which results in

$$T(z) J^a(w) \sim \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} \tag{2.82}$$

Remark. Once again, the geometry of the Lie algebra makes itself visible and forces us to make corrections to what we would usually expect. The geometry is taken into account here by changing the expected $\frac{1}{2k}$ into $\frac{1}{2(k+h^\vee)}$.

Now that we have fixed what we would like to be an energy-momentum tensor, let's show it actually is one. Showing that the modes of T form the Virasoro algebra is sufficient to prove that this tensor is an energy-momentum one, and proving as a by-product that the theory is indeed a 2D CFT. But the commutation relations of the modes of an operator can be read from its OPE with itself. We therefore only have to show that the OPE $T(z)T(w)$ looks like what we expect from an energy-momentum tensor. We have

$$T(z)T(w) = \frac{1}{4\pi i(k + h^\vee)} \oint \frac{\mathrm{d}x}{x-w} (T(z) J^a(x) J^a(w) + T(z) J^a(x) J^a(w)) \tag{2.83}$$

Inserting (2.82) in this, we have

$$\begin{aligned} T(z)T(w) = &\frac{1}{4\pi i(k + h^\vee)} \oint \frac{\mathrm{d}x}{x-w} \left(\frac{J^a(x)}{(z-x)^2} + \frac{\partial J^a(x)}{z-x} \right) J^a(w) \\ &+ \frac{1}{4\pi i(k + h^\vee)} \oint \frac{\mathrm{d}x}{x-w} J^a(x) \left(\frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} \right) \end{aligned} \tag{2.84}$$

Then summing over all generators of $\mathfrak{g}$, we finally have

$$T(z)T(w) \sim \frac{k \dim \mathfrak{g}}{2(k + h^\vee)(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{2.85}$$

Wich is exactly the form we would expect, showing the modes of T indeed form the Virasoro algebra and the WZW-model indeed is a 2D CFT. From this, we read the central charge of the model

$$c \equiv \frac{k \dim \mathfrak{g}}{k + h^\vee} \tag{2.86}$$

This construction of the energy-momentum tensor is called the Sugawara construction.

To finish this section, we may compute the commutation relations of the Virasoro algebra with the affine Kac-Moody algebra. The modes of the energy momentum tensor read

$$\begin{aligned} L_n &= \frac{1}{2(k+h^\vee)}(: J^a J_a :)_n \\ &= \frac{1}{2(k+h^\vee)}\left(J^a_n J^a_0 + \sum_{m \neq 0} \left[J^a_m J^a_{n-m} + J^a_{n-m} J^a_m \right] \right) \end{aligned} \tag{2.87}$$

With an extensive use of the normal ordering, we may find

$$[L_n, J^a_m] = -m J^a_{n+m} \tag{2.88}$$

This shows explicitely that any affine Kac-Moody algebra contains a Viraso algebra naturally embedded in its universal envelopping algebra. Thinking about it the other way around, this also gives a general non-trivial extension of the Virasoro algebra for any simple Lie algebra. Moreover, such extensions can be extended to semisimple Lie algebras, by taking a sum of WZW-models on simple Lie algebras. These facts should already hint at the importance of the WZW-model. We will try to understand better the role of these models in the study of 2D CFTs in the next subsection.

2.11 Representations of the algebra

To understand better the role of WZW-models and affine Kac-Moody algebras in the study of 2D CFTs, let's consider the WZW-model of level k associated to a simple Lie algebra $\mathfrak{g}$ and let's take a closer look at its space of states.

The Kac-Moody algebra $\hat{\mathfrak{g}}_k$ acts on the Hilbert space of the theory, hence all states will transform in representations of $\hat{\mathfrak{g}}_k$. We can then decompose the space of state in a sum of representations of $\hat{\mathfrak{g}}_k$, as in the general case where the space of states decomposes in a sum of representations of the Virasoro algebra. Let's consider a representation $\mathcal{M}$ of $\hat{\mathfrak{g}}_k$ in the space of state.

As for the case of representations of the Virasoro algebras seen in the previous lecture notes, a physical representation must have its energy bounded from below. We therefore have an eigenstate $|\lambda\rangle$ with eigenvalue e such that its eigenvalue is the lowest inside $\mathcal{M}$. But we have seen previously that

$$[L_0, J^a_m] = -m J^a_m \tag{2.89}$$

which leads to

$$\begin{aligned} L_0 J^a_m |\lambda\rangle &= (J^a_m L_0 + [L_0, J^a_m])|\lambda\rangle \\ &= (e - m) J^a_m |\lambda\rangle \end{aligned} \tag{2.90}$$

From this, we deduce that for all $n > 0$, we must have

$$J_n^a|\lambda\rangle = 0 \tag{2.91}$$

Remark. States obeying these conditions are called Kac-Moody primary states, or more generally primary states when the context is clear. They are the exact analog of the Virasoro primary states, and generate the representations of the affine Kac-Moody algebra. Kac-Moody primary states $|\lambda\rangle$ are generated by Kac-Moody primary fields φ^λ, where the condition of being Kac-Moody primary becomes

$$J^a(z)\varphi^\lambda(w) \sim \frac{J_0^a\varphi^\lambda(w)}{z-w} \tag{2.92}$$

But how can we label highest-weight modules? We have that the zero-modes $(J_0^a)_a$ commute together as the elements of $\mathfrak{g}$. So a highest-weight representation of $\mathfrak{g}$ is included in $\mathcal{M}$, and we can choose $|\lambda\rangle$ to be the highest-weight state of this representation. This subrepresentation in facts completely determines $\mathcal{M}$. We can thus choose λ to label this subrepresentation, labeling the whole module as a consequence. We call $\mathcal{M}_\lambda$ the module associated to the subrepresentation λ. We can then decompose the space of states as

$$\mathcal{H} = \bigoplus_\lambda \mathcal{M}_\lambda \tag{2.93}$$

An important theorem [Per15, p. 89] is that for any affine Kac-Moody algebra $\hat{\mathfrak{g}}_k$, there exist only a finite number of λ such that $\mathcal{M}_\lambda$ is unitary and irreducible. Hence, the sum (2.93) is always finite, and WZW-models are rational 2D CFTs.

We cannot emphasize enough the importance of this result. We have seen in 1.2 that studying RCFTs boils down to finding non-trivial extensions of the Virasoro algebra having a finite number of representation. The affine Kac-Moody is just that, for any simple Lie algebra. What's more is that we can "add" such extensions of simple Lie algebras as much as we want, by considering the sum of the associated theories.

Semisimple Lie algebras are a very general and broad class of structures, able to express any symmetry. We can therefore have very big and general extensions of the Virasoro algebra using affine Kac-Moody algebras. But then, constraining the theory behind these algebras and partially gauge fixing WZW-model, we can get an even broader class of extensions of the Virasoro algebra, generating a very broad class of RCFTs. As we will see, these extensions have the good property of not containing any null field for generic values of the central charge. It is theorized that all RCFTs whose chiral algebra doesn't contain any null field can be obtained by reducing a WZW-model.

Chapter 3

Casimir algebras

In this section, we derive the historically first kind of $\mathcal{W}$-algebra found after affine Kac-Moody algebra, which can be derived by reducing the WZW-model.

3.1 Classical Toda field theories

To understand Casimir algebras, we first need to introduce properly Toda field theories. Toda field theories are 2D CFT associated to semisimple Lie algebras along with a few external parameters, in the same fashion as WZW-models. We will introduce them following the original work of Leznov and Saveliev.

As with WZW-models, we want to define a nice model on a semisimple Lie algebra $\mathfrak{g}$. Our objective is to study RCFTs and extended Virasoro algebras. Therefore, by nice, we mean a conformally invariant, solvable model taking into account the structure of the Lie algebra. To make it solvable, we will try to make an integrable model.

We want to take into account the structure of the Lie algebra $\mathfrak{g}$. Let's give a few notations for it. We write r the rank of $\mathfrak{g}$, and write $\Phi \subset \mathbb{R}^r$ the set of roots of $\mathfrak{g}$. We will usually write $\alpha \in \Phi$ a root. Moreover, we write $\Delta = \{\alpha_i\}$ a set of simple roots. When using the subscript i, we will often imply that $i \in \mathbb{N}_r$. For each of these simple roots, we associate an element H_i of the Cartan subalgebra. The other generators of $\mathfrak{g}$ are labelled by roots, and written E_α. As the generators associated to simple roots will be of special interest to us, we also write $E_i^\pm \equiv E_{\pm\alpha_i}$.

The set of simple roots gives a basis for Φ, where the coordinates are either all positives or all negatives. This allows to divide the roots in two parts, those with positive coordinates Φ^+ and those with negative coordinates Φ^-. In turn, this allows us to define 2 subalgebras of $\mathfrak{g}$, which are

$$\mathfrak{g}_\pm = < (H_i)_i, (E_\alpha)_{\alpha \in \Phi^\pm} > \tag{3.1}$$

We are now ready to define the object of study. As with the WZW-model, we would like to consider (massless) particles moving on $\mathfrak{g}$. Moreover, as with the WZW-model, since we want to consider a CFT, these particles will be massless and will first move on the worldsheet, which will itself move on $\mathfrak{g}$. Therefore, we will once again have left and right moving particles.

We would like to consider the momentums of the left and right moving particles. We therefore consider a two components field (depending on the chiral coordinates)

$$\mathcal{A}(z, \bar{z}) = \big(\mathcal{A}(z, \bar{z}), \bar{\mathcal{A}}(z, \bar{z})\big) \in (\mathfrak{g}^2)^{\mathbb{R}^2} \tag{3.2}$$

where $\mathcal{A}(z, \bar{z})$ corresponds to the momentum of the right moving particles whilst $\bar{\mathcal{A}}(z, \bar{z})$ corresponds to the particle of the left moving particles.

Since we want these fields to describe the momentum of particles, we should put as a constraint that there should be a field $g : \mathbb{R}^2 \to G$ actually describing the particles as in the WZW-model, where G is the Lie group associated to $\mathfrak{g}$. Given the field g, what are the chiral momentums of the particles? It should be some form of derivative of the field in the left and right coordinates, pulled back to 0 as to be well defined on $\mathfrak{g}$. Looking back at the momentum of particles on a Lie group found in 2.4, we see that a good definition of the momentum can be the Maurer-Cartan form, which also take into account the symmetry of the Lie structure. We will take the opposite of the Maurer-Cartan form, though it is only a matter of convention

$$\begin{aligned} \mathcal{A} &= g\partial\ g^{-1} \\ \bar{\mathcal{A}} &= g\bar{\partial}\ g^{-1} \end{aligned} \tag{3.3}$$

Remark. Note that this is different from the WZW-model, where the momentum of the left and right moving particles have a different expression. In particular, in the WZW-model, the momentum of the right moving particles takes the form $J = \partial g\ g^{-1}$

We also want to reflect the decoupling of the particles in left and right moving parts on the Lie algebra $\mathfrak{g}$. To do so, we also define a notion of "left" and "right" on the Lie algebra, by assigning a different "direction" to the left and right moving particles. In practice, we add the following constraint

$$\begin{aligned} \mathcal{A} &\in \mathfrak{g}_+ \\ \bar{\mathcal{A}} &\in \mathfrak{g}_- \end{aligned} \tag{3.4}$$

We may want g to be smooth, sufficiently to assume that

$$\partial\bar{\partial}g = \bar{\partial}\partial g \tag{3.5}$$

In this case, (3.3) implies that (and is locally equivalent to)

$$[\partial + \mathcal{A}, \bar{\partial} + \bar{\mathcal{A}}] = 0 \tag{3.6}$$

which is a condition of integrability for the system, as desired.

Now, let's precise (3.4). It is not necessary to have both right and left moving particles be able to move along the Cartan subalgebra, which acts as a neutral place. Therefore, we may choose without loss of generality to have only the right moving particles moving along the the Cartan subalgebra. This can be described by a field $\Psi = (\Psi_i)_i : \mathbb{R}^2 \to \mathbb{R}^r$, directly representing the particles projected onto the Cartan subalgebra. We may express their momentum as

$$P_{\text{Cartan}} = \sum_i \partial\Psi_i H_i \tag{3.7}$$

We may also arbitrarily choose to only have particles move along simple roots and their opposite counterparts, for simplicity. Then, the remaining momentum should be generated by the (E_i^+) for $\mathcal{A}$, and by the (E_i^-) for $\bar{\mathcal{A}}$. We are only interested in the difference between the two, as we can always renormalize the $(E_i^\pm)$ without loss of generality. Therefore, we could set

$$\begin{aligned} \mathcal{A} &= P_{\text{Cartan}} + \sum_i E_i^+ \\ \bar{\mathcal{A}} &= \sum_i \varphi_i E_i^- \end{aligned} \tag{3.8}$$

for some field $\varphi = (\varphi_i)_i : \mathbb{R}^2 \to \mathbb{R}^r$.

We will modify this in 3 ways, as to have a prettier model in the end. First, we add an external coupling parameter γ to the momentum along the $(E_i^\pm)$, as to understand the way it is coupled with the momentum P_{Cartan}. Second, as φ should describes some kind of energy of the system, we expect some kind of Boltzmann distribution, and therefore put φ in an exponential. Third, as usually done in statistical physics, we also add another external coupling constant β in front of φ as to represent some kind of temperature of the system. We rewrite

$$\begin{aligned} \mathcal{A} &= \sum_i \left(\partial\Psi_i H_i + \gamma E_i^+\right) \\ \bar{\mathcal{A}} &= \sum_i \gamma e^{\beta\varphi_i} E_i^- \end{aligned} \tag{3.9}$$

As motivated, we see that γ controls the momentum on the roots. The bigger γ is, the more momentum there is along the roots (compared to along the Cartan subalgebra). On the other hand, we see that β controls the difference of momentums between the left and right moving particles. If $\varphi_i > 0$, then the sign of β also controls wether there is more momentum along E_i^+ or E_i^-, with a bigger momentum along the latter for $\beta > 0$.

We want Ψ and φ to obey the constraints mentionned earlier, and in particular the integrability condition (3.6). Still supposing enough smoothness to have $\partial\bar{\partial} = \bar{\partial}\partial$, inserting the expression for the fields (3.9), we get

$$\begin{aligned} 0 &= \partial\bar{\mathcal{A}} - \bar{\partial}\mathcal{A} + \mathcal{A}\bar{\mathcal{A}} - \bar{\mathcal{A}}\mathcal{A} \\ &= \sum_i \left(\gamma\beta\partial\varphi_i e^{\beta\varphi_i} E_i^- - \partial\bar{\partial}\Psi_i H_i + \gamma^2 e^{\beta\varphi_i} H_i\right) - \sum_{i,j} \gamma\partial\Psi_i e^{\beta\varphi_j} k_{j,i} E_j^- \end{aligned} \tag{3.10}$$

where we have used the known commutation relations of the $(E_i^\pm)$ and (H_i), and where $(k_{i,j})$ is the Cartan matrix of $\mathfrak{g}$. Projecting this equation onto the (H_i), we have

$$\Psi_i = \beta \sum_j k_{i,j}^{-1} \varphi_j \tag{3.11}$$

whilst projecting onto the (E_i^-), we get

$$\partial\bar{\partial}\Psi_i = \gamma^2 e^{\beta\varphi_i} \tag{3.12}$$

Putting these two equations together, we get the classical generalised Toda field equation

$$\partial\bar{\partial}\varphi_i = \frac{\gamma^2}{\beta}\sum_j k_{i,j} e^{\beta\varphi_j} \tag{3.13}$$

Identifying the different spaces $\mathbb{R}^r$ as $\varphi = \sum_i \varphi_i \alpha_i$, the Toda field equation can be rewritten in term of vectors

$$\partial\bar{\partial}\varphi = \frac{\gamma^2}{\beta}\sum_i \alpha_i e^{\beta\alpha_i\cdot\varphi} \tag{3.14}$$

Remark. This is the most general Toda field equation. We have added the external parameters γ and β as is done by some authors. Note that some other authors may keep $\gamma = 1$ or $\beta = 1$

We can make up a Lagrangian whose equation of motion gives (3.14)

$$\mathcal{L} = \partial\varphi\bar{\partial}\varphi + \frac{\gamma^2}{\beta^2}\sum_{i=1}^{r} e^{\beta\alpha_i\cdot\varphi} \tag{3.15}$$

This Lagrangian defines the Toda field theory. We should emphasize the natural form of this Lagrangian, considering it describes massless particles. It is made up of a first kinetic term, equal to $(\partial^\mu)^2\varphi$ in the usual (x,t) coordinates. It is then followed by a potential term, enforcing the constraints we have set on our theory similarly to a Lagrange multiplier, making sure that the direction of the particles is coupled with the sign of the (positive or negative) simple roots. We easily see here how γ controls the momentum on the roots (if $\gamma = 0$, there is no potential) and how β disperses the particles.

3.2 Symmetries

In the following discussion, we will suppose without loss of generality that $\alpha = 1$. We have introduced Toda field theory by saying we were looking for a solvable conformal invariant model on Lie algebras. The model we then defined is solvable, and takes into account the geometry of Lie algebras. The question one should now ask is: is the theory conformally invariant?

To answer this, let's take a conformal transformation $(z,\bar{z}) \to (f(z), \bar{f}(\bar{z}))$. Under this transformation, the field φ changes as

$$\varphi(z,\bar{z}) \to \varphi(f(z),\bar{f}(\bar{z})) + \frac{1}{\beta}\rho\ln(f'(z)\bar{f}'(\bar{z})) \equiv \varphi'(z,\bar{z}) \tag{3.16}$$

where ρ is the sum of the fundamental weights of $\mathfrak{g}$. But the Toda field equation (3.14) is invariant under this transformation, meaning φ' is still a solution of the equation. Therefore, the theory is invariant under conformal transformation.

Knowing this, we can look at how the field transform under an infinitesimal transformation. We will only look at the holomorphic dimension. For an holomorphic infinitesimal transformation $z \to z + \epsilon(z)$, the field changes by

$$\delta_\epsilon \varphi = \epsilon \partial \varphi + \frac{1}{\beta} \rho \partial \epsilon \tag{3.17}$$

From this, one can read the associated energy-momentum field

$$T = \partial \varphi \cdot \partial \varphi - \frac{2}{\beta} \rho \cdot \partial^2 \varphi \tag{3.18}$$

Now that we know the classical Toda field theory is invariant under conformal transformations, we want to know if the theory has other symmetries. Indeed, if it has other symmetries, and if the other symmetries do not decouple completely from the conformal symmetry, then the algebra of symmetries will form a $\mathcal{W}$-algebra.

We should now emphasize that this theory is a classical theory, where we are not taking into account any quantum effect whatsoever. Therefore, the algebra of symmetries would form what we call a classical $\mathcal{W}$-algebra, and not a quantum $\mathcal{W}$-algebra as defined earlier in the chapter.

Let's try to find some symmetries. We may try to find symmetries by exploiting properties of the system. We know our system decouples into right and left moving particles. Let's concentrate first on symmetries in the holomorphic part.

We consider the fundamental representation of $\mathfrak{g}$. This representation possesses a highest weight vector, which can be found by summing all $(E_i^+)_i$. We write its dual $\langle w|$. Because we have constrained the particles moving left to move on $\mathfrak{g}_-$, recalling that their momentums is given by $\bar{\mathcal{A}}$, we have

$$\begin{aligned} \bar{\mathcal{A}} \in \mathfrak{g}_- &\Rightarrow \langle w|\bar{\mathcal{A}} = 0 \\ &\Rightarrow \langle w|\bar{\mathcal{A}} g(z,\bar{z}) = 0 \\ &\Rightarrow \bar{\partial}\langle w|g(z,\bar{z}) = 0 \end{aligned} \tag{3.19}$$

Such that defining $\chi = \langle w|g(z,\bar{z})$, χ is a function only of z. From there, we should treat every different types of simple Lie algebra differently. We will only give the general idea of the construction, though a detailed construction for each type of simple Lie algebra can be found in [MS91]

We can decompose χ in some double-indexed basis (viewing the elements of $\mathfrak{g}$ as matrices, we decompose χ in the natural basis of $M_n(\mathbb{R})$), which turns (3.19) in a serie of equations in which we can introduce $\mathcal{A}$. From these equations, one may build an operator $\Gamma(\mathcal{A})$, the Lax operator associated to the fundamental representation of $\mathfrak{g}$, acting on $\mathfrak{g}$, which reduces all of the equations found above in

$$\Gamma(\mathcal{A})\chi = 0 \tag{3.20}$$

We should emphasize the fact that $\Gamma(\mathcal{A})$ embodies the constraint that left moving particles should move along $\mathfrak{g}_-$. As this is a constraint enforced on the whole system, we find as we can expect

$$[\Gamma, \bar{\partial}] = 0 \tag{3.21}$$

Therefore, Γ is a conserved quantity, and generates symmetries of the Toda field theory, symmetries associated to the constraint on the left moving particles. The algebra made of these generators is the so called classical $\mathcal{W}\mathfrak{g}$-algebra, and is thus part of the complete algebra of symmetries of the theory.

Similarly, we can derive the same result from the antiholomorphic constraints, considering $|w\rangle$ instead. Therefore, another decoupled classical $\mathcal{W}\mathfrak{g}$-algebra appears in the complete algebra of symmetries of the theory.

Are these all of the symmetries of the theory? Let's recall how we derived the Toda field equation. We started with two fields $\mathcal{A}$ and $\bar{\mathcal{A}}$, representing the momentum of right and left moving particles. We then asked for the two fields to obey

$$[\partial + \mathcal{A}, \bar{\partial} + \bar{\mathcal{A}}] = 0 \tag{3.22}$$

which is locally equivalent to asking for the existence of a field g such that $\mathcal{A}$ and $\bar{\mathcal{A}}$ are indeed the right and left momentums of excitations on g.

Then, we asked the fields $\mathcal{A}$ and $\bar{\mathcal{A}}$ to take the specific form (3.9). With this specific form, we derived the Toda field equation from (3.22).

One can show that (3.22) is invariant under transformations of the form

$$\begin{aligned} \mathcal{A}(\varphi) &\to \mathcal{A}(\varphi) + [\partial + \mathcal{A}(\varphi), \omega] \\ \bar{\mathcal{A}}(\varphi) &\to \bar{\mathcal{A}}(\varphi) + [\bar{\partial} + \bar{\mathcal{A}}(\varphi), \bar{\omega}] \end{aligned} \tag{3.23}$$

for arbitrary ω. However, this kind of transformation does not necessarily preserve the form of the fields (3.9). Knowing that the Toda field equation is equivalent to the double constraint from (3.9) and (3.22), we understand that the symmetries of the Toda field theory are exactly the transformations of the form (3.23) which preserve the form of the pair of fields.

These symmetries form a space. We will not dive into the heavy computations needed to find this space, which must be done individually for each type of simple Lie algebra. The detail of the computations can be found in [MS91]. The main result is that the dimension of this space corresponds exactly to the number of generators found from the holomorphic and antiholomorphic Lax operators described above. Therefore, the symmetries generated by the Lax operators are precisely the symmetries of this system.

We conclude that the algebra of symmetries associated to the classical Toda field theory defined on $\mathfrak{g}$ is made up of two decoupled copies of the classical $\mathcal{W}\mathfrak{g}$-algebra. The next subsections will try to get the same result in a quantum version of the classical Toda field theory.

3.3 Quantum Toda field theory

We now want to quantize the whole theory to make a quantum field theory, in hope of obtaining a serie of quantum $\mathcal{W}$-algebras, the quantum $\mathcal{W}\mathfrak{g}$-algebras. To do so, we use canonical methods for quantizing fields.

First, fields can only interact with each other if they are not space-like separated, from Wightman's axioms. We will consider therefore be driven to consider light cones, that is in 2 dimensions light rays. Without loss of generality, we will assume the light rays originate from $x = 0$. Given a time τ, we will then write $(\mathcal{T}_\tau, \bar{\mathcal{T}}_\tau)$ the two light rays originating from $(\tau, 0)$, that is

$$\mathcal{T}_\tau = z = \frac{1}{2}\tau, \bar{z} \geq \frac{1}{2}\tau, \quad \bar{\mathcal{T}}_\tau = \bar{z} = \frac{1}{2}\tau, z \geq \frac{1}{2}\tau \tag{3.24}$$

We write $\varepsilon(x)$ the sign function, such that $\varepsilon(\mathbb{R}_+^*) = \{1\}$, $\varepsilon(\mathbb{R}_-^*) = \{-1\}$, and $\varepsilon(0) = 0$. With these, the commutation relation of a r scalar components field $(\varphi_i)_{i \in \mathbb{N}_r}$ taken in two distinct points of a single light ray writes

$$\begin{aligned} [\varphi(z, \bar{z}), \varphi(w, \bar{z})] &= -\frac{1}{4} i\hbar\delta_{ij}\varepsilon(z - w) \\ [\varphi(z, \bar{z}), \varphi(z, \bar{w})] &= -\frac{1}{4} i\hbar\delta_{ij}\varepsilon(\bar{z} - \bar{w}) \end{aligned} \tag{3.25}$$

Moreover, we set an arbitrary mass[1] m which will describe the particles of our theory. This way, we can set the energy of a particle depending on its momentum using the relativistic dispersion relation

$$m^2 = \omega(p)^2 - p^2 \tag{3.26}$$

We write $\mathbf{p}(p) = (\omega(p), p)$ the energy-momentum of a particle. With all of these, we can define an annihilation operator, which creates an "anti" excitation of momentum p on the field φ_j along some arbitrary light-cone $(\mathcal{T}_\tau, \bar{\mathcal{T}}_\tau)$

$$\begin{aligned} a_j(p) = &\int_{\mathcal{T}_\tau} i\mathrm{d}\bar{z} \ \left(e^{i\mathbf{p}\cdot(t,x)}\bar{\partial}\varphi_j - \bar{\partial}e^{i\mathbf{p}\cdot(t,x)} \ \varphi_j \right) \\ &+ \int_{\bar{\mathcal{T}}_\tau} i\mathrm{d}z \ \left(e^{i\mathbf{p}\cdot(t,x)}\partial\varphi_j - \partial e^{i\mathbf{p}\cdot(t,x)} \ \varphi_j \right) \end{aligned} \tag{3.27}$$

and we can compute its commutation relations using (3.25)

$$\begin{aligned} [a_i(p), a_j(q)] &= 0 \\ [a_i(p), a_j^\dagger(q)] &= 4\pi\hbar\omega(p)\delta_{ij}\delta(p - q) \end{aligned} \tag{3.28}$$

We can then construct the Fock space $\mathcal{F}_{(\mathcal{T}_\tau, \bar{\mathcal{T}}_\tau)}$ on which these operators act, by defining a vacuum state $|0\rangle$ such that

$$a_i(p)|0\rangle = 0 \quad \forall i, p \tag{3.29}$$

and by successively applying creation operators of the form $a_i^\dagger(p)$ for arbitrary p and i.

By inverting (3.27), one can find an expression for φ on $(\mathcal{T}_\tau, \bar{\mathcal{T}}_\tau)$

$$\begin{aligned} \varphi_j &= \phi_j + \bar{\phi}_j, \quad \phi_j = \bar{\phi}_j^\dagger \\ \phi_j &= \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{4\pi\omega(p)} e^{-i\mathbf{p}\cdot(t,x)} \end{aligned} \tag{3.30}$$

In this formula, we see the difference between the "creation" part $\bar{\phi}_i$ and the "annihilation" part ϕ_i of the field φ_i, made of creation operators $a_i^\dagger(p)$ and

[1] We know a theory must describe massless particles for it to be conformal. And indeed, the mass will diseappear later on to preserve conformal invariance.

annihilation operators $a_i(p)$ respectively. The operation of normal ordering, necessary in any quantum field theory to prevent one from getting something from the vacuum, thus places the ϕ_i to the right of the $\bar{\phi}_i$. This explains the particular importance of the commutator of those two fields. For $\mathbf{x}, \bar{\mathbf{x}}$ on the same light ray, we have

$$[\phi_i(\mathbf{x}), \bar{\phi}_i(\bar{\mathbf{x}})] = \delta_{ij}\hbar\Delta(\mathbf{x} - \bar{\mathbf{x}}) \tag{3.31}$$

with

$$\Delta(\mathbf{x}) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{4\pi\omega(p)} e^{-i\mathbf{p}\cdot\mathbf{x}} \tag{3.32}$$

However, this diverges for $\mathbf{x}\cdot\mathbf{x} \to 0$. We should therefore add a lower and upper energy bound to the theory, as is usually done in quantum field theory. We define

$$\phi_j^{\mathrm{reg}} = \int_{-\Lambda_1}^{\Lambda_2} \frac{\mathrm{d}p}{4\pi\omega(p)} e^{-i\mathbf{p}\cdot(t,x)} \tag{3.33}$$

and similarly for $\bar{\phi}_j^{\mathrm{reg}}$, which gives the commutator

$$\Delta^{\mathrm{reg}}(\mathbf{x}) = \int_{-\Lambda_1}^{\Lambda_2} \frac{\mathrm{d}p}{4\pi\omega(p)} e^{-i\mathbf{p}\cdot\mathbf{x}} \tag{3.34}$$

Now that we have quantized the fields, we must adapt the classical Toda field equation. We recall from (3.13) that they can take the form

$$\partial\bar{\partial}\varphi_i = \frac{\gamma^2}{\beta} \sum_j k_{i,j} e^{\beta\varphi_j} \tag{3.35}$$

The most straightforward way of adapting this would be to simply normal-order the exponential, as to have vanishing vacuum expectation values. However, as we have seen from the discussion earlier, taking the normal ordering of this exponential would lead to the apparition of an exponential of Δ. This is problematic, as Δ depends in its definition of ω, which itself depends on the mass of the particles. We have chosen an arbitrary mass to determine the creation and annihilation operators, but the resulting quantum model shouldn't depend on mass knowing the original classical model doesn't.

Let's compute exactly how the mass appears. We know that for any A, B, supposing $[A, B]$ commutes with A and B, we have

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \tag{3.36}$$

We can then compute that for a transformation of mass $m \to \mu$, the normal ordered exponential changes as

$$e^{\beta\bar{\varphi}_i} e^{\beta\varphi_i} \to e^{\beta\bar{\varphi}_i} e^{\beta\varphi_i} e^{\frac{1}{2}\beta^2\hbar(\Delta_m(0) - \Delta_\mu(0))} \tag{3.37}$$

where Δ_m and Δ_μ corresponds to Δ taken with mass m and μ respectively. We know that $\Delta(\mathbf{x})$ is divergent when $\mathbf{x} \to 0$. However, we can compute

$$\lim_{\Lambda_1,\Lambda_2\to\infty} (\Delta_m^{\mathrm{reg}}(0) - \Delta_\mu^{\mathrm{reg}}(0)) = \frac{1}{4\pi} \ln(\frac{\mu^2}{m^2}) \tag{3.38}$$

such that the normal ordered exponential transforms as

$$e^{\beta\bar{\varphi}_i}e^{\beta\varphi_i} \to e^{\beta\bar{\varphi}_i}e^{\beta\varphi_i}\frac{\mu}{m}^{\frac{1}{4\pi}\beta^2\hbar} \tag{3.39}$$

Therefore, we see that

$$m^{-\frac{1}{4\pi}\beta^2\hbar} : e^{\beta}\varphi_j : \tag{3.40}$$

does not depend on m. Writing the above factor as σ, we can define the quantum Toda field equation as

$$\partial\bar{\partial}\varphi_i = \frac{\gamma^2\sigma}{\beta}\sum_j k_{i,j} : e^{\beta\varphi_j} : \tag{3.41}$$

3.4 The underlying quantum algebra

We now want to study the quantum algebra behind the symmetries of this theory. We remember from 3.2 that the symmetries of the theory are classically generated by the Lax operator. We thus want to adapt the Lax operator to the quantum theory.

Once again, we won't go into the technical details as they are different for each type of simple Lie algebra. The details can be found in [MS91]. We will also once again consider the holomorphic dimension. We remember that we constructed the Lax operator from a serie of equations by putting one after the other a sequence of operators each embodying one of the equations, such that the serie of equations could take the form

$$\Gamma(\mathcal{A})\chi = 0 \tag{3.42}$$

We will write this sequence of operators $(b_i)_{i\in\mathbb{N}_n}$ such that writing the dependance in z, we have

$$\Gamma(z) = b_n(z)\dots b_2(z)b_1(z) \tag{3.43}$$

In a quantum theory, we want everything to be normal ordered. However, it might not be trivial to compute the normal-ordered Lax operator. We instead use Wick's theorem, which gives a relation between normal ordered operators and time ordered operators. This reduces the problem to finding the time-ordered Lax operator, which we will do juste now.

To make sure the Lax operator is time ordered, instead of taking each of the $(b_i)_i$ at z, we should take them at $(z+i\varepsilon)$ as to forcibly time-order this operator. We write the resulting new Lax operator Γ_ε.

$$\Gamma_\varepsilon(z) = b_n(z+n\varepsilon)\dots b_2(z+2\varepsilon)b_1(z+\varepsilon) \tag{3.44}$$

Then, Wick's theorem says that

$$: \Gamma_\varepsilon := \Gamma_\epsilon - \sum_{(b_i,b_j)} b_n \dots \overbracket{b_i \dots b_j} \dots b_1 \tag{3.45}$$

From this, one may compute $[\bar{\partial}, : \Gamma_\varepsilon :]$ in term of $[\bar{\partial}, \Gamma_\varepsilon]$ and a sum of complicated but computable terms. On the other hand, one can also compute $[\bar{\partial}, \Gamma_\varepsilon]$ using the quantum Toda field equation. One can then obtain an expression for $[\bar{\partial}, :\Gamma_\varepsilon :]$ for which we can take ϵ to 0. Then, one can finally obtain that

$$[\bar{\partial}, :\Gamma_0 :] = 0 \Leftrightarrow \gamma = \beta + \frac{1}{\beta} \tag{3.46}$$

Remark. The holomorphic (quantum) energy-momentum field is then

$$T =: \partial\varphi \cdot \partial\varphi : -2\left(\frac{\rho^\vee}{\beta} + \beta\rho\right) \cdot \partial^2\varphi \tag{3.47}$$

We notice that in comparison with the classical energy-momentum field (3.18), quantum corrections have added a $-2\beta\rho\cdot\partial^2\varphi$ term, and have turned the already present ρ into its dual $\rho^\vee$.

We know the form γ has to take from (3.46). In fact, the real condition is a bit more flexible, as it takes the form of a linear equation. But linear equations have the particularity that the sum of 2 solutions is still a solution. This leads to considering the sum of 2 Toda field equations, and to defining the new following model

$$\partial\bar{\partial}\varphi = \sum_i \alpha_i \left(: e^{\beta\alpha_i\cdot\varphi} : + : e^{-\frac{1}{\beta}\alpha_i\cdot\varphi} :\right) \tag{3.48}$$

for which one can prove that we have $[\bar{\partial}, :\Gamma_0 :] = 0$.

This new equation defines what is known as the conformally extended Toda field theory. Thanks to their conservations, the holomorphic normal-ordered Lax operator generates the holomorphic part of the symmetries of this theory, whilst the antiholomorphic one generates the remaining of the symmetry. But each of these Lax operator form the quantum $\mathcal{W}\mathfrak{g}$-algebra, also known as the Casimir algebra associated to the Lie algebra $\mathfrak{g}$. In particular, we know that a Virasoro algebra is contained in any Casimir algebra. This proves that the theory contains 2 chiral Virasoro algebras among its symmetries, thus proving the theory is conformally invariant at the quantum level.

The study of conformally extended Toda field theories has been of great importance in the study of $\mathcal{W}$-algebras, because it generates a serie of $\mathcal{W}$-algebras but also because it provides the Lagrangian formulation for a minimal model associated to each of these algebras, that is a theory where the only set of symmetries is the $\mathcal{W}$-algebra. Finding a Lagrangian to describe a minimal model of a $\mathcal{W}$-algebra is highly non-trivial.

3.5 The WZW-model perspective

Up until now, we have derived a serie of non-trivial $\mathcal{W}$-algebras, which are very interesting in their own right. However, their study is quite limited and we would now like to continue our quest for new $\mathcal{W}$-algebras. The question is therefore: can we use Casimir algebras to derive new $\mathcal{W}$-algebras? Can we generalise them? And to this question, the answer is yes thanks to a key insight found in [For+89]. In this paper, the authors showed that one can approximately[2] see Toda field theories as constrained WZW-models. This point of view then allows

[2] This approximation should be precised later on.

one to see Casimir algebras as constrained affine Kac-Moody algebras, raising the question of wether one can generalise this kind of constraint.

Let's first recall some of the results we found on WZW-models in 2. The WZW-model is a 2D CFT defined for a given Lie algebra $\mathfrak{g}$ and level k by the Lagrangian (2.50), describing particles moving on a string itself living in the Lie group G associated to the Lie algebra $\mathfrak{g}$. As seen in 2.4 and 2.8, this theory has a $G(z) \times G(\bar{z})$ symmetry, whose conserved currents correspond to the momentums of the left and right moving particles, given by

$$\begin{aligned} J(z) &= J^a(z)T_a = k\partial g\; g^{-1} \\ \bar{J}(\bar{z}) &= \bar{J}^a(\bar{z})T_a = kg^{-1}\bar{\partial}g \end{aligned} \tag{3.49}$$

We can rewrite these currents in term of the generators of $\mathfrak{g}$ given in the previous section, as

$$\begin{aligned} J &= \sum_{\alpha_i \in \Delta} \left(J_i H_i + J_i^+ E_i^+ + J_i^- E_i^-\right) + \sum_{\alpha \in \Psi^+ \setminus \Delta} \left(J_\alpha^+ E_\alpha + J_\alpha^- E_{-\alpha}\right) \\ \bar{J} &= \sum_{\alpha_i \in \Delta} \left(\bar{J}_i H_i + \bar{J}_i^+ E_i^+ + \bar{J}_i^- E_i^-\right) + \sum_{\alpha \in \Psi^+ \setminus \Delta} \left(\bar{J}_\alpha^+ E_\alpha + \bar{J}_\alpha^- E_{-\alpha}\right) \end{aligned} \tag{3.50}$$

where the dependance in z and $\bar{z}$ is implicit. The equations of motion of the WZW-model write

$$\bar{\partial}J = 0, \qquad \partial\bar{J} = 0 \tag{3.51}$$

Instead of decomposing the momentums on a basis $(T_a)_a$ of $\mathfrak{g}$, we can give them a dependance in the Lie algebra through the trace operation

$$\begin{aligned} J(\lambda)(z) &= \mathrm{Tr}(\lambda \cdot J(z)) \\ \bar{J}(\lambda)(\bar{z}) &= \mathrm{Tr}(\lambda \cdot \bar{J}(\bar{z})) \end{aligned} \tag{3.52}$$

Remark. We must notice that this decomposition of the momentum is dual to the decomposition we previously had, with regards to the trace. For $\lambda \in \mathfrak{g}$, $J(\lambda)$ corresponds to J^{λ^*} where λ^* is the dual of λ through the trace. Effectively, we are switching the momentum along a root with the momentum along the opposite of this root, thus switching $\mathfrak{g}_+$ with $\mathfrak{g}_-$.

Now, the main result is that for arbitrary numbers $(\mu^i)_i, (\nu^i)_i \subset \mathbb{R}_+^*$, if we constrain the momentums of the left and right moving particles in the WZW-model by saying

$$\begin{aligned} J(E_{\alpha_i}) = k\mu^i, \quad \bar{J}(E_{-\alpha_i}) = -k\nu^i \qquad &\text{for } \alpha_i \in \Delta \\ J(E_\alpha) = 0, \quad \bar{J}(E_{-\alpha}) = 0 \qquad &\text{for } \alpha \in \Phi^+ \setminus \Delta \end{aligned} \tag{3.53}$$

Then the theory reduces (locally) to a Toda field theory.

Remark. We can rewrite (3.53) using the decomposition (3.50) as

$$\begin{aligned} J_i^- = \frac{1}{2}k|\alpha_i|^2\mu^i, \quad \bar{J}_i^+ = -\frac{1}{2}k|\alpha|^2\nu^i \qquad &\text{for } \alpha_i \in \Delta \\ J_\alpha^- = 0, \quad \bar{J}_\alpha^+ = 0 \qquad &\text{for } \alpha \in \Phi^+ \setminus \Delta \end{aligned} \tag{3.54}$$

This kind of result should feel somewhat intuitive. Indeed, both the WZW-model and the Toda field theory describe massless particles moving on a string moving on a Lie group. Moreover, both models have the particles separated in 2 groups, those moving left and those moving right. The difference is that in a Toda field theory, the particles moving left move only along $\mathfrak{g}_-$ whilst the particles moving right move only along $\mathfrak{g}_+$. But the constraints above fix the movement of the particles moving left along $\mathfrak{g}_+$ and fix the movement of the particles moving right along $\mathfrak{g}_-$ in the WZW-model, thus allowing them to freely move only on the part they would be able to move in a Toda field theory. However, we see that everything is not fixed by these constraints. In fact, there is a residual gauge redundancy left when using the constraints above, which leads to the locality of the result. This phenomenom is discussed in more details in [Bal+90].

To show that the WZW-model indeed reduces to a Toda field theory, we first need to use the local Gauss decomposition.

Remark. We should insist that this decomposition only works locally in a general semisimple Lie algebra. It cannot be extended to the whole algebra. For this reason, the reduction of the WZW-model to a Toda field theory is only local. Global solutions of a reduced WZW-model may not lead to global solutions of the associated Toda field theory

Given the field of study in the WZW-model g, Gauss's decomposition says that we can locally write

$$g = ABC \tag{3.55}$$

where

$$\begin{aligned} A &= \exp\left(\sum_{\alpha\in\Phi^+} a_\alpha E_\alpha\right) \\ B &= \exp\left(\sum_{\alpha_i\in\Delta} \phi_i H_i\right) \\ C &= \exp\left(\sum_{\alpha\in\Phi^+} c_\alpha E_{-\alpha}\right) \end{aligned} \tag{3.56}$$

Let's look at the holomorphic current J first. Using this decomposition, we can rewrite it as

$$\begin{aligned} J &= k\partial(ABC)\ C^{-1}B^{-1}A^{-1} \\ &= k\partial A\ A^{-1} + A\partial B\ B^{-1}A^{-1} + AB\partial C\ C^{-1}B^{-1}A^{-1} \end{aligned} \tag{3.57}$$

Using the derivative of exponentials and the commutation relations of the $(H_i)_i$ and $(E_\alpha)_\alpha$, we can see that the term $[\partial A\ A^{-1}]$ contributes to the $(J^+_\alpha)_{\alpha\in\Psi^+}$, the term $[A\partial B\ B^{-1}A^{-1}]$ contributes to the $(J^+_\alpha)_{\alpha\in\Psi^+}$ and $(H_i)_{\alpha_i\in\Delta}$, whilst the term in $[AB\partial C\ C^{-1}B^{-1}A^{-1}]$ contributes to the $(J^-_\alpha)_{\alpha\in\Psi^+}$ and $(H_i)_{\alpha_i\in\Delta}$.

But (3.53) only constrains the $(J^-_\alpha)_{\alpha\in\Psi^+}$, such that we may discard the two first terms. Moreover, the contribution from this term to the $(H_i)_{\alpha_i\in\Delta}$ is generated by the commutation of A with $B\partial C\ C^{-1}B^{-1}$, and this commutation only generates contributions to $(H_i)_{\alpha_i\in\Delta}$. Therefore, the constraint directly writes out

$$kB\partial C\ C^{-1}B^{-1} = \frac{1}{2}k\sum_{\alpha_i\in\Delta}|\alpha_i|^2\mu^i E_i^- \tag{3.58}$$

Or equivalently

$$\partial C\ C^{-1} = \sum_{\alpha_i \in \Delta} \frac{1}{2} |\alpha_i|^2 \mu^i E_i^- e^{\frac{1}{2}\sum_{\alpha_j \in \Delta} k_{ij}\phi_j} \tag{3.59}$$

Similarly, doing the same kind of analysis for $\bar{J} = k\ C^{-1}B^{-1}A^{-1}\bar{\partial}(ABC)$, we find that the condition on the antiholomorphic current given in (3.53) rewrites

$$A^{-1}\bar{\partial}A = \sum_{\alpha_i \in \Delta} \frac{1}{2} |\alpha_i|^2 \nu^i E_i^+ e^{\frac{1}{2}\sum_{\alpha_j \in \Delta} k_{ij}\phi_j} \tag{3.60}$$

Looking at the equation of motion for the holomorphic current, we see

$$0 = \bar{\partial}\left(k\partial A\ A^{-1} + A\partial B\ B^{-1}A^{-1} + AB\partial C\ C^{-1}B^{-1}A^{-1}\right) \tag{3.61}$$

But inserting (3.59) and (3.60), assuming that all fields are smooth enough to freely permute ∂ and $\bar{\partial}$, we can remove all fields A and C, such that we get an equation only on (ϕ_i). The same happens for the other equation of motion on the antiholomorphic current.

Finally, putting everything together, we can obtain the equivalent equations

$$\partial\bar{\partial}\phi_i + \frac{1}{2}|\alpha_i|^4 \mu^i \nu^i e^{\frac{1}{2}\sum_{\alpha_j \in \Delta} k_{ij}\phi_j} = 0 \tag{3.62}$$

which is equivalent to the Toda field equation (3.14) for a general Toda field theory where $\beta = 2$, where we have decomposed ϕ into its components in an orthonormal basis $(e_i)_i$ instead of the basis given by the simple roots (α_i), and where we have given a dependance to the external parameter γ in i such that $\gamma_i = |\alpha_i|^2 \mu^i \nu^i$. We see that the bigger μ^i or ν^i is, the bigger the momentum on the roots will be in the equivalent Toda field theory, which seems logical.

Chapter 4

The general Drinfield-Sokolov reduction

We have seen how one can reduce WZW-models to obtain Casimir algebras, which are a kind of $\mathcal{W}$-algebras. In this section, we will generalise this reduction in order to obtain a broader class of $\mathcal{W}$-algebras. We refer to the annex A for any reader incomfortable with Hamiltonian reductions and the BRST procedure.

4.1 General constraints

We consider the WZW-model of level k on a semisimple Lie group G. As was done in 3, we first consider the classical theory, which is simpler to deal with. We will consider the quantum theory afterwards. We want to reduce the WZW-model in the most general way possible, by imposing constraints on the momentums $J(z)$ and $\bar{J}(\bar{z})$, respectively of the right and left moving particles.

We will only consider the holomorphic part of the theory (particles moving right) since we know the same process will apply to the other part. Since the model describes particles moving on G, the momentum $J(z)$ is an element of $\mathfrak{g}$. How can we impose constraints on it? We would like to divide $\mathfrak{g}$ into 2 complementary parts such that it is fixed on one of the parts, and free on the other. But first, we need to be able to decompose $J(z)$ onto a basis of $\mathfrak{g}$ in an expressible way. To do so, we use the trace operator as an inner scalar product of $\mathfrak{g}$

$$\langle u, v\rangle = \mathrm{Tr}(u \cdot v) \qquad \forall u, v \in \mathfrak{g} \tag{4.1}$$

With this, we can choose a subalgebra $\Gamma \subset \mathfrak{g}$ on which we want to fix the momentum, meaning we want to fix $\langle \Gamma, J\rangle$. We also set M, to which we want our momentum to correspond on Γ. Then, the momentum J should take the form

$$J(z) = M + j(z) \qquad \text{with} \quad j(z) \in \Gamma^{\perp} \tag{4.2}$$

where $\Gamma^{\perp}$ is the subspace orthogonal to Γ with regards to the trace.

Remark. Actually, we are not fixing J along Γ but along its dual with regards to the trace. The intuition is the same, but it allows for a better expression of the constraint. In particular, we directly have that $\Gamma^\perp$ is a complement of the dual of Γ with regards to the trace.

The formulation (4.2) allows for an easy understanding of the constraint we are considering. But we can equivalently reformulate it as a function which should vanish, as it is usually done in Lagrangian or Hamiltonian mechanics with constrainted systems, as discussed in A.3. To do so, we simply define a function measuring the distance between J and M on (the dual of) Γ:

$$\phi_\gamma(z) = \langle \gamma, J(z) \rangle - \langle \gamma, M \rangle = 0 \qquad \forall \gamma \in \Gamma \tag{4.3}$$

We should note that there is some ambiguity in the definition of M. In particular, M can be arbitrarily shifted along $\Gamma^\perp$, as the constraint only fixes J to M on the dual of Γ. As such, we can suppose without loss of generality that M vanishes on $\Gamma^\perp$.

(4.2) can describe any kind of constraint put on the momentum J. However, in the general case, we do not know how to impose it, or wether it is even possible. We should therefore put some conditions on the choice of Γ and M, to only keep constraints we know how to enforce. Following [Feh+92], we will suppose that the constraints we want to enforce are all first-class constraints. This means that for any $\alpha, \beta \in \Gamma$, we should have in the reduced theory

$$\{\phi_\alpha, \phi_\beta\} = 0 \tag{4.4}$$

where the Poisson bracket comes from the full theory. This assumption is actually not so strong, and covers most of the known reduced WZW-models. It also makes sense physically, as we are interested in "proper" reductions, were the constraints really modify the dynamics of the system.

Let's now see what this first-classness implies. We have derived the quantum structure of the current algebra of the WZW-model in (2.71). At a classical level, the Lie bracket becomes a Poisson bracket, and the bracket of the components of the momentums can be written at equal time using the trace and delta functions as

$$\{\langle u, J(x)\rangle, \langle v, J(y)\rangle\}|_{x^0=y^0} = \langle [u,v], J(x)\rangle \delta_1(x-y) + k\langle u, v\rangle \delta_1'(x-y) \tag{4.5}$$

where δ_1 is the spatial delta function, meaning $\delta_1(x-y) = \delta(x^1 - y^1)$ for $x = (x^0, x^1)$ in the usual (non-chiral) coordinates. Having the bracket of the momentums at equal time is sufficient, since we are only considering quantities depending on the holomorphic dimension, which can therefore be moved along the antiholomorphic dimension to be at equal times. Using this to compute the bracket of the constraints (4.3), we get

$$\{\phi_\alpha(x), \phi_\beta(y)\} = \phi_{[\alpha,\beta]}(x)\delta_1(x-y) + \langle M, [\alpha,\beta]\rangle \delta_1(x-y) + \langle \alpha, \beta\rangle \delta_1'(x-y) \tag{4.6}$$

It is interesting to note that in this relation, the dependance in the level k diseappear, meaning the way we are bypassing the geometry of G doesn't matter for the "properness" of the constraints. (4.4) holds if and only if each of the terms in the above expression vanish weakly, individually meaning

$$\begin{aligned} [\Gamma, \Gamma^\perp] &\subset \Gamma^\perp \\ [M, \Gamma] &\subset \Gamma^\perp \\ \Gamma &\subset \Gamma^\perp \end{aligned} \tag{4.7}$$

Given the second condition, we see that the choice of a valid M is reduced to

$$M \in [\Gamma, \Gamma]^{\perp} / \Gamma^{\perp} \tag{4.8}$$

intuitively meaning that we can give the momentum of the particles a non-zero fixed part only as a counterpart (dual) of the residue of the free part. This residue always exists, as the last condition implies that Γ must be solvable. Note that the last condition can be satisfied by choosing Γ nilpotent, although the nilpotency is not necessary.

4.2 Ensuring conformal invariance

Imposing a constraint of the form 4.2 breaks the classical conformal symmetry of the WZW-model. Indeed, if M is non-zero (and isn't in $\Gamma^{\perp}$), some components of the momentum are fixed to non-zero constants, which stay constant through rescales of the system. However, the momentum is supposed to have conformal dimension 1, meaning all components of the momentum should scale linearly under rescales of the system. The question is then to know wether the reduced model, where the constrained surface has been gauged out, is once again conformally invariant or not. For it to be again conformally invariant, the action of the conformal group on the WZW-model phase space needs to be modified as to leave the constrained surface invariant.

Therefore, the question is: how can we modify the action of the conformal group to make it leave the constrained surface invariant? We know that the action of the (holomorphic) conformal group action is encoded in the energy-momentum field $T_{\mathrm{WZW}}(z)$ of the WZW-model, which gives the energy of the theory. We may thus change the landscape of the question, to ask: how can we modify the energy of the system to take into account the reduction of the theory?

To answer this, we should take a step back. We want to prevent the particles from going into certain directions, or more generally we want to modify the general behaviour of free particles as to prevent them from going into certain directions. We can see this as removing certain directions from the phase space, or more generally as modifying the phase space as to prevent free particles from going into certain directions. This kind of modification can be achieved by curving the space in which the particles live, as to create with the worldsheet some sort of funnel for them to stay in.

This way, we can see a link with general relativity. The main idea of general relativity is to curve the space to modify the behaviour of free particles. In Einstein's picture, mass curves the space as to attract other objects to it. In our case, we could put some very very large fictive mass along the path the particles can take, such that they are funneled into this path, effectively implementing the reduction of the phase space.

From this perspective, we can then understand what form should the modification of the energy $T_{\mathrm{WZW}}(z)$ take. We want to add a very very large fictive mass, creating a gravitational potential to capture the particles and force them to go in some directions. In fact, we do not need the fictive mass, but only the gravitational potential generated from it. We want to be in the approximation

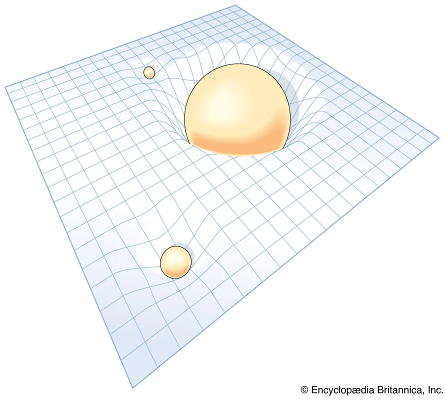

Figure 4.1: Illustration of general relativity

where the fictive mass generating the potential is so big that it is not affected by external forces, and never moves, much like how we consider gravity on earth. On earth, we assume the gravitational field is constant and equal to $\vec{g}$, such that the potential energy generated by gravity is equal to

$$E_p = m\vec{g} \cdot \vec{z} \tag{4.9}$$

with m the mass of the object subject to the gravitational field, and z its height (its distance to the ground). In our theory, we would like to have something similar, and write

$$T_{\text{red}}(z) = T_{\text{WZW}}(z) + E_p \tag{4.10}$$

with E_p some sort of potential energy generated from the fictive mass. The question is then, how to replace m, $\vec{g}$ and $\vec{z}$ in (4.9)? Let's look at the case of gravity on earth once again. The gravitational field is assumed to be constant, equal to $\vec{g}$. It thus exert a force equal to $m\vec{g}$ on the objects. But according to Newton's first law,

$$m\vec{g} = \vec{F} = m\vec{a} = m\dot{\vec{v}} = \dot{\vec{p}} \tag{4.11}$$

such that we can replace $m\vec{g}$ by the derivative of the momentum $J'(x)$. On the other side, $\vec{z}$ should be some measure of the distance from the direction we want the particles to stay in, and can just be an element $-H \in \mathfrak{g}$ for now. With these, we then rewrite the energy-momentum field

$$T_{\text{red}}(z) = T_{\text{WZW}}(z) - \langle H, J'(z) \rangle \tag{4.12}$$

Without giving big interpretations of H, we can already see it will only affect the compontents of J' and thus of J along a complement of $J^{\perp}$. Therefore, the geometry of the space will be modified only on a complement of $J^{\perp}$, inting at the fact that we should have $J^{\perp} = \Gamma^{\perp}$.

Now, let's actually compute wether the modification we made to the energy-momentum field actually corrected the action of the conformal group on the WZW-model phase space, and what are the conditions imposed on H for it to work. To do so, let's consider an infinitesimal (holomorphic) conformal transformation $z \to z - f(z)$. We have $\delta_f z = -f(z)$, such that assuming T_{red} is the energy-momentum field of the theory,

$$\begin{aligned}\delta_f J(z) &= -\int \mathrm{d}y f(y)\{T_{\text{red}}(y), J(z)\} \\ &= f(z)J'(z) + f'(z)\left(J(z) + [H, J(z)]\right) + f''(z)H\end{aligned} \tag{4.13}$$

Injecting the constraint (4.2) in this, we get

$$\delta_f j(z) = f(z)j'(z) + f'(z)\left(j(z) + [H, j(z)] + [H, M] + M\right) + f''(z)H \tag{4.14}$$

Now, we know the issue with the previous action of the conformal group was that it didn't leave the constraints invariant. For our modification to repair the conformal invariance of our system, T_{red} should leave the constrained surface invariant, meaning it shouldn't affect the components of the momentum along the trace dual of Γ, meaning we should have $\delta_f j(z) \in \Gamma^\perp$. For it to be true, we must have

$$\begin{aligned} H &\in \Gamma^\perp \\ [H, \Gamma^\perp] &\subset \Gamma^\perp \\ ([H, M] + M) &\in \Gamma^\perp \end{aligned} \tag{4.15}$$

Reciprocally, if there a $H \in \Gamma^\perp$ satisfying these conditions, then we know how to modify the energy-momentum field to get a new working energy-momentum field for the reduced theory, proving the reduced theory is conformally invariant. Note however that the non-existence of such an element doesn't imply a priori that the theory is not conformally invariant. It only means that we haven't found the right way to modify the energy-momentum field.

From now on, we will assume that our choice of M and Γ is such that we can find an element H satisfying the above conditions. This assumption is not automatic, but should cover a sufficiently large spectrum of reductions, since modifying the energy-momentum tensor by an element H is a very natural and general way to do it as explained above.

Let's look at the last condition imposed on H, that is $([H, M] + M) \in \Gamma^\perp$. Since $M \notin \Gamma^\perp$, we understand that H should be chosen to compensate for M. We could then expect that H only compensate for M, without creating further shifts in $\Gamma^\perp$. This leads us to considering a H such that $[H, M] + M = 0$. When can we choose such H?

To answer this, let's look back at the intuition behind H. H should describe a potential vector field on $\mathfrak{g}$, associating to each $X \in \mathfrak{g}$ a vector pointing in the direction of $\Gamma^\perp$, the space of directions along which the particles should be free. But since $\Gamma^\perp$ is nothing more than a linear subspace, H should be able to decompose $\mathfrak{g}$ in a basis of proper elements $(X_a)_a$, where the vector associated to each X_a is in the direction of X_a itself. We can thus safely assume that H is diagonalizable, giving a grading on $\mathfrak{g}$. But if H is diagonalizable, we can always shift it by a constant such that we have $[H, M] + M = 0$. This further hints at the fact that H should be diagonalizable.

From now on, we will therefore always assume that H is diagonalizable, and that $[H, M] = -M$. Note that H then induces a grading on $\mathfrak{g}$. For any eigenvalue m of H, we will write $\mathfrak{g}_m$ the associated eigenspace.

4.3 The classical $\mathcal{W}$-algebra

We consider the WZW-model defined on the Lie algebra $\mathfrak{g}$ at level k. Let (Γ, M) a pair defining a reduction of the WZW-model, and H the associated element verifying the conditions described above. We now want to study the algebra behind the reduced theory. The theory comes down to the dynamics of the momentums J and $\bar{J}$, commuting with each other. The algebra we want to study is thus the algebra formed by the components of each momentum. But since the 2 momentums commute, their components will commute too, forming 2 distinct algebras. We will therefore restrict our study to the algebra of the components of a single momentum, say J.

The constraints we are imposing on the theory is that $J(z)$ must take the form

$$J(z) = M + j(z) \qquad \text{with} \quad j(z) \in \Gamma^{\perp} \tag{4.16}$$

The algebra is thus generated by the components of $j(z)$.

Since ad_H leaves Γ and $\Gamma^{\perp}$ invariant, we can decompose these spaces in a graded basis. Let $(X^i_m)_{i,m \in I \times \mathbb{R}}$ be a graded basis of $\Gamma^{\perp}$, such that for any $(i, m) \in I \times \mathbb{R}$ we have $[H, X^i_m] = m X^i_m$. Then we can decompose $j(z)$ in this basis, and write

$$j(z) = \sum_{i,m \in I \times \mathbb{R}} j^i_m(z) X^i_m \tag{4.17}$$

Now, let's recall 4.14. Using the decomposition above and the fact that $[H, M] = -M$, we have

$$\delta_f j^i_m(z) = f(z) j^i_m(z)' + f'(z)\,(1 + m)\, j^i_m(z) + f''(z) H \tag{4.18}$$

But we know from the previous lecture notes that a Virasoro primary field φ of conformal dimension h should transform under conformal transformations as

$$\delta_f \varphi = f(z)\partial\varphi + f'(z) h \varphi \tag{4.19}$$

Therefore, we see that the algebra of the theory is generated by the fields $(j^i_m(z))_{i,m}$ which are almost primary with conformal dimension $(1 + m)$, with the exception of the H term.

We see what is the reduced algebra behind the reduced theory. But in this form, the algebra is not very computable. Moreover, there is a lot of gauge redundancies in the theory, since the theory is invariant under a local action of Γ. We should therefore try to fix the form of the current $J(z)$ in a *normal* form through gauge transformations, such that the algebra takes a nice form.

What would be the right normal form? Inspired by (4.6), we might want to define the 2-form

$$\omega_M(\alpha, \beta) = \langle M, [\alpha, \beta] \rangle \qquad \forall \alpha, \beta \in \mathfrak{g} \tag{4.20}$$

such that the second condition of (4.7) ensuring the primarity of the constraints rewrites

$$\omega_M(\alpha, \beta) = 0 \qquad \forall \alpha, \beta \in \mathfrak{g} \tag{4.21}$$

We may then want to define a graded subset $\Theta \subset \mathfrak{g}$ such that Θ would be dual to Γ with respects to ω_M, and ask the current $J(z)$ to be in the form [1]

$$J_{\text{red}}(z) = M + j_{red}(z) \qquad \text{with} \quad j_{red}(z) \in \Gamma^{\perp} \cap \Theta^{\perp} \tag{4.22}$$

This constraint effectively restricts the form of the current. But is it really nothing more than a gauge fixing? In other words, are we not further reducing the model by imposing this constraint? To show that this constraint is actually a gauge fixing constraint, we may explicitely give for any current $J(z)$ an associated gauge transformation such that the transformation brings $J(z)$ to the form (4.22). We do not know how to do this in the general case. However, under some very natural assumptions, this transformation can be expressed polynomially using an algorithm found in [Feh+92]. Following the authors of [Feh+92], we will therefore call a reduction obeying these assumption a *polynomial* reduction. Note however that not all reductions are polynomial. Considering a polynomial reduction only allows for a better form of the algebra.

The first obvious assumption we should make is being able to take Θ a dual of Γ under ω_M. But to take its dual, Γ should be a sympleptic space under ω_M. Let's analyse ω_M. We write $\mathcal{K}_M = \text{Ker}(\text{ad}_m)$, and set $\mathcal{T}_M$ a complementary space to $\mathcal{K}_M$ such that $\mathfrak{g}$ decomposes in a direct sum as

$$\mathfrak{g} = \mathcal{K}_M \oplus \mathcal{T}_M \tag{4.23}$$

With these notations, seeing using the linearity and cyclicity of the trace that

$$\langle [a,b], c\rangle = \langle [c,a], b\rangle \tag{4.24}$$

we find that $\omega_M(\mathcal{K}_M, \mathfrak{g}) = 0$, and that ω_M is non-degenerate on $(\mathcal{T}_M, \mathcal{T}_M)$. Therefore, to be able to take Θ a dual of Γ, we should have $\Gamma \subset \mathcal{T}$, or more generally

$$\Gamma \cap \mathcal{K}_M = \{0\} \tag{4.25}$$

This is our first additional condition.

Now, before actually discussing of the algorithm, let's reformulate the constraint (4.22). Recall how we reformulated the original constraint (4.2) as a function which must be annihilated (4.3), similarly to Lagrange multipliers. We can once again use the same strategy, and reformulate (4.22) as

$$\chi_\theta(z) = \langle J(z), \theta\rangle - \langle M, \theta\rangle = 0, \qquad \forall \theta \in \Theta \tag{4.26}$$

We should also write the general form of a gauge transformation

$$g(z) = e^{\sum_{h,l} a_h^l(z) \gamma_h^l} \tag{4.27}$$

which acts on the constrained current as

$$j(z) \to j^g(z) = e^{a\cdot\gamma}(j(x) + M)e^{-a\cdot\gamma} + \left(e^{a\cdot\gamma}\right)' e^{-a\cdot\gamma} - M \tag{4.28}$$

The objective of the algorithm is thus to provide a gauge transformation $g(z)$ such that $j^g(z)$ answers $\chi_\theta(z) = 0$.

[1] This form is often called the Drinfield-Sokolov gauge, as it was developped by analogy with their work

Before diving into the computations, we should set a basis (γ_h^l) of Γ and a dual basis (θ_k^j) of Θ, such that the grade of γ_h^l under H is h, with l labelling the multiplicity of the elements of the basis with grade h. Note that with these, θ_k^j has grade $(1-k)$ under H.

With these notations, let's try to understand (4.28). To do so, we will first consider graded gauge transformations, generated by the local action of $\mathfrak{g}_h$ for fixed h. Let $h \in \mathbb{R}$ a grade. We consider a transformation of the form $g_h(z) = e^{\sum_l a^l(z)\gamma_h^l}$. We should also set $\Theta_h = \langle\ \theta_k^l\ /\ k \leq h\ \rangle$.

Let's consider the effect of (4.28) on $\chi_\theta(z)|_{\theta\in\Theta_h}$. First, the derivative term does not contribute, since $\langle \gamma_h^l, \theta_k^j \rangle = 0$ for $k < h$. Let's now look at the effect of $j(z)$. We have

$$e^{a_h\cdot\gamma_h} j(z) e^{-a_h\cdot\gamma_h} = j(x) + [a_h(z)\cdot\gamma_h, j(z)] + \dots \tag{4.29}$$

It would simplify things greatly to consider only the first term and remove the effects of the gauge transformation. But noticing that every other term is the commutator of something along γ_h (which is of grade h) with something in $\Gamma^\perp$, and recalling (4.24), we see that the effects of the gauge transformation diseappear on Θ_h if $[\mathfrak{g}_h, \Theta_h] \in \Gamma$. But knowing that the degree of the bracket of two elements is the difference between their degrees plus one, we see that $[\mathfrak{g}_h, \Theta_h] \subset \mathfrak{g}_{\geq 1}$. Therefore, we may impose

$$\mathfrak{g}_{\geq 1} \subset \Gamma \tag{4.30}$$

to make the commutators diseappear. This is our second additional condition.

Now, we can also use the fact that the two basis (γ_h^l) and (θ_k^j) are dual one to the other and the identity (4.24) to compute the terms with an M, which putting everything together yields

$$\langle \theta_k^i, j^{g_h}(z) \rangle = \langle \theta_k^i, j(x) \rangle - a_h^i(z)\delta_{hk} \qquad \forall k \leq h \tag{4.31}$$

But by identification, we see that

$$\langle \theta_k^i, j(x) \rangle = 0 \tag{4.32}$$

is equivalent to

$$\langle \theta_k^i, j^{g_h}(x) \rangle = 0 \qquad \text{for} \quad k < h \tag{4.33}$$

and by setting

$$a_h^i(z) = \langle \theta_h^i, j(x) \rangle \tag{4.34}$$

we can fix

$$\langle \theta_h^i, j^{g_h}(x) \rangle = 0 \qquad \text{for} \quad k = h \tag{4.35}$$

Therefore, we may define the family (a_h^i) recursively following (4.34), such that the gauge parameter $a_h^i(z)$ always has a closed expression polynomial in the current, and such that eventually we have $\chi_\theta(z) = 0$.

Now, let's try to properly understand the reason for each assumption. First, we assumed that $\Gamma \cap \mathcal{K}_M = \{0\}$. This assumption is necessary to be able to take the dual of Γ through ω_M, and to define the normal form the current should take. But intrinsically, what is this form? We can notice that the space $\mathcal{V} = \Gamma^\perp \cap \Theta^\perp$

to which we constrain the momentum is graded, and is the complementary of the image of Γ through ad_M (which is injective on Γ by assumption). Therefore, we have the direct sum decompositon

$$\Gamma^\perp = [M,\Gamma] + \mathcal{V} \tag{4.36}$$

Written like this, we can see the momentum in its normal form as constrained on $\Gamma^\perp$ (as prescribed by (4.2)) but gauged out of Γ[2], and therefore constrainted to the small interstice between $\Gamma^\perp$ and Γ.

In fact, this perspective should be the right one, since for any $\mathcal{V}$ complementary to $[M,\Gamma]$ in $\Gamma^\perp$, we can choose a Θ dual to Γ such that $\mathcal{V} = \Gamma^\perp \cap \Theta^\perp$. Therefore, the data of Θ is equivalent to that of $\mathcal{V}$.

Now, let's consider the second assumption. Its statement is equivalent to saying that

$$\Gamma^\perp \subset \mathfrak{g}_{>-1} \tag{4.37}$$

In fact, this statement is very natural, to the point we could almost take it as an axiom and never consider reductions which do not verify this. Indeed, as we saw at the beginning of this section (4.18), the algebra associated to the model is made up of the components of $j(z)$, which are all primary fields of conformal dimension $(1+h)$ where m is the grade of the component. However, we have seen in the previous lecture notes how the conformal dimension dictates the behaviour of the field under rescales. In particular, a field will loose energy when it is zoomed on if and only if its conformal dimension is positive, whilst the inverse will happen if its conformal dimension is negative. This implies that the conformal dimension of any physical field must be positive. But $j(z)$ is constrained to $\Gamma^\perp$, which means that we should have $(1+h)$ positive for any graded component of $j(z)$, meaning we should have (4.37).

Remark. Note that this assumption also implies that M is uniquely determined, since no element of $\Gamma^\perp$ is of grade -1 and can shift M.

Finally, let's look at what we can get by combining the 2 assumptions. Using the two assumptions along with the conditions put on Γ, we get that Γ can only contain positive grades

$$\Gamma \subset \mathfrak{g}_{>0} \tag{4.38}$$

which also implies that

$$\mathfrak{g}_{\geq 0} \subset \Gamma^\perp \tag{4.39}$$

This clearly shows the freedom left by the pair (M,H) in choosing Γ: given a grading and a nilpotent element, there exist a small number of different choices for Γ, resulting in different physical reductions of the theory.

$$\mathfrak{g}_{\geq 1} \subset \Gamma \subset \mathfrak{g}_{>0} \qquad \mathfrak{g}_{\geq 0} \subset \Gamma^\perp \subset \mathfrak{g}_{>-1} \tag{4.40}$$

4.4 Quantum affine $\mathcal{W}$-algebras

We have derived the classical affine $\mathcal{W}$-algebras known to mathematicians, through the imposition of constraints which are themselves constrained by a

[2]Actually, gauged out of the injective image of Γ in $\Gamma^\perp$ through ad_M

variety of conditions. But how do we quantize this algebra, in a way that is usable? The answers comes in 2 parts. First, how do we choose the right Γ, M, H, Θ?

The solution in front of such a complex data subject to so many conditions is to constrain further the data, as to be able to summarize the conditions in a few words. In particular, the known approach is to only consider a good pair (M, H), whose conditions are discussed in appendix B. Given such a good pair, we can choose to have

$$\Gamma = \mathfrak{g}_{>0} \qquad \Gamma^{\perp} = \mathfrak{g}_{\geq 0} \tag{4.41}$$

which yields a valid reduction. Therefore, we can associate a polynomial reduction of the WZW-model to any good pair of the underlying Lie algebra. We usually write the pair (f, x) such that f describes the constant shift of the momentum of the particles, whilst x describes the potential keeping the particles along some $\Gamma^{\perp}$.

But then, how do we quantize the algebra? We should first make the theory quantum. We have already looked at the quantum version of the WZW-model, which has very small nuances. However, these nuances won't affect the final result. Therefore, we should simply directly apply the BRST procedure. The BRST procedure as described in the annex A can be applied straightforwardly to our Lagrangian multiplier (4.3), as to get a quantum affine $\mathcal{W}$-algebra as known by mathematicians.

Actually, someome attentive enough might notice that the algebra we obtain by using the BRST procedure with the Lagrangian multiplier (4.3) doesn't exactly correspond to the algebra described in [DK06]. Indeed, this paper considers more general kind of WZW-models, where the underlying Lie algebra may actually be a Lie superalgebra. In this more general case, M can be either even or odd. Our method works if M is even, but breaks for odd M. To solve this issue, [KRW03] introduced the addition of another ghost to add the possibility of M being odd. It is this enhanced $\mathcal{W}$-algebra that is now often considered by mathematicians.

Appendix A

Hamiltonian reductions

In this appendix, we give a quick tour of the different methods to quotient out the symmetries of a physical system. We review the classical Hamiltonian reduction, and another algebraic prodecure allowing for the same reduction, namely the BRST procedure.

A.1 The space of configurations

We consider a physical system, which is described by a point in its space of configurations. The space of configurations forms a differentiable manifold M, the space in which the system lives. For exemple, in classical non-relativistic mechanics, M may be $\mathbb{R}^6$ with coordinates (x, y, z, p_x, p_y, p_z) where the configuration of the system takes into account the position and momentum of the system.

From classical mechanics, we know that the position and momentum of a system are very important. The configuration of any system[1] should at least contain some sort of position q and momentum p, both making the phase space (q, p) of the system. Therefore, we should be able to know what is q or p on M, meaning we should be able to equip M with an additional structure as to take into account the position and momentum.

In the case where M is a flat vector space, this is quite easy. The right additional structure is a Poisson bracket, turning M into a Poisson manifold, classically defined for any two functions $F, G \in C^\infty(M)$ by

$$\{F, G\} = \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial G}{\partial q}\frac{\partial F}{\partial p} \tag{A.1}$$

This structure allows one to think about Hamiltonian mechanics in a very convenient way. Indeed, let H be the Hamiltonian of the system

$$H : M \to \mathbb{R} \tag{A.2}$$

H is a function giving the energy of a system according to its configuration. The Euleur-Lagrange equation, equivalent to Newton's first law, can be reformulated

[1]Living freely in a continuous space. As we are interested in continous symmetries, we won't talk about systems defined on lattices

using the Hamiltonian as

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H}{\partial p}, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial q} \tag{A.3}$$

which are known as Hamilton's equations. Notice how the two equations are dissymmetric, setting a difference between q and p. Using the Poisson bracket defined on $C^\infty(M)$, we can rewrite these equations as

$$\dot{q} = \{q, H\}, \quad \dot{p} = \{p, H\} \tag{A.4}$$

which first of all is nice looking, but also symmetrizes the role of p and q. Furthermore, for any quantity of the system $F(q, p)$ depending on the position and/or momentum of the system, we have

$$\dot{F} = \{F, H\} \tag{A.5}$$

In particular, we have $\{H, H\} = 0$, which is nothing but the law of energy conservation. In a more general fashion, for any quantity F, the fact that $\{F, H\} = 0$ means F is a quantity conserved along the motion of the system.

Furthermore, the Jacobi identity tells us that if we have two functions F, G such that $\{F, H\} = 0$ and $\{G, H\} = 0$, then we also have $\{\{F, G\}, H\} = 0$, meaning the Poisson bracket of two conserved quantities is still a conserved quantity. The Poisson manifold structure on M therefore induces a Poisson algebra structure on the space of all conserved quantities of the system.

Geometrically, we can easily interpret the Poisson bracket on $C^\infty(M)$. To any smooth function H on M, the Poisson bracket gives a derivation $\{\_, H\}$, which can be seen as a vector field ξ_H. Interpreting H as a function giving the energy of the system depending on its configuration (interpreting it as a Hamiltonian), we see that the associated vector field ξ_H gives for any point in the configuration space the direction in which the system evolves.

However, M is not always flat and it is not always possible to define a Poisson manifold in this straightforward manner. For exemple, when studying a pendulum, the position is only defined modulo 2π, such that M is an infinite cylinder. How does one generalise this Poisson bracket? Instead of defining a Poisson bracket

$$\{\_, \_\} : C^\infty(M) \times C^\infty(M) \to \mathbb{R} \tag{A.6}$$

we should consider a more general 2-form

$$\omega : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathbb{R} \tag{A.7}$$

A 2-form doesn't exactly takes the same arguments as a Poisson bracket. However, supposing it is non-degenerate, it gives a mapping between vectors and 1-forms, such that as with the Poisson bracket, it gives a mapping between smooth functions and vector fields by

$$\xi_F \equiv \omega^{-1}(dF, \cdot) \qquad \forall F \in C^\infty(M) \tag{A.8}$$

Then, we can make the link between the Poisson bracket and the 2-form by

$$\{F, G\} = \omega(\xi_F, \xi_G) \tag{A.9}$$

We have seen how one can translate a Poisson bracket into a 2-form. But in a general space, what are the constraints one should put on ω for it to describe an actual physical space of configurations? A Poisson bracket must be bilinear, antisymmetric, and generates derivatives on $C^\infty M$. A 2-form is also bilinear and antisymmetric, but how does one translate the condition of generating derivatives? The answer is by asking ω to be a closed form.

Therefore, we can equip the space of configurations M of any physical system with a closed non-degenerate 2-form ω. (M,ω) is then called a sympleptic manifold, and is the subject of study of a whole area of mathematics.

A.2 Symmetries in this space

Now, let's look at how symmetries are represented on this sympleptic manifold. M usually posseses symmetries. As an exemple, if M is the position-momentum space in 3 dimensions (for a total of 6 dimensions), then M is symmetric under translations in space and under a shift of momentum. This doesn't mean that these are symmetries of the considered theory, but only encodes the structure of the space of configuration. On the other side, any symmetry of the system considered should also be a symmetry of the space of configurations.

We know that symmetries are encoded by (connected) Lie groups. We therefore dispose of the Lie group G of symmetries of M, along with an action of G on M. If the theory itself has symmetries, it is encoded as a (connected) subgroup $H \subset G$ of G also acting on M by the induced action of G.

Deriving the action of G on M, we get a map from $\mathfrak{g}$ to the set $\mathscr{X}(M)$ of vector fields on M.

$$\begin{aligned} \mathfrak{g} &\to \mathscr{X}(M) \\ g &\to \xi_g \end{aligned} \tag{A.10}$$

Inuitively, given a "direction of transformation" g, this map transmits the movement from $\mathfrak{g}$ to M, by giving the direction in which this transformation would go for each point of M. In the case of a symmetry by spatial translations, g can for exemple be a direction in which the system can translate, resulting in ξ_g being a field of vectors pointing in this direction.

We should now discuss of the interactions between the action of G and the sympleptic structure. We will give a few definitions, allowing us to determine the "quality" of the action. To do so, we will write ι the contraction operator, such that for a n-form ω and a vector field ξ, we have

$$\iota(\xi)\omega(\eta_1, \dots, \eta_{n-1}) = \omega(\xi, \eta_1, \dots, \eta_{n-1}) \tag{A.11}$$

For a vector field ξ, we will also write ξ^b its associated 1-form, given by $\xi^b = \iota(\xi)\omega$

First, we want to qualify how "natural" is a vector field. We are discussing of Hamiltonian mechanics, and we are considering vector fields as directions in which the system should evolve. Therefore, we want to know if a vector field ξ can originate from some potential (or energy) function H, such that if the energy of the system is given by H then the direction in which it will evolve is given by ξ, i.e. $\xi_H = \xi$. We thus say a vector field ξ is *sympleptic* if ξ^b is closed, and we say it is *hamiltonian* if ξ^b is exact. Note that the exactness is exacly what we want: if a vector field ξ is hamiltonian, we can associate to it a function $\phi_\xi \in C^\infty(M)$ such that

$$\xi^b + \mathrm{d}\phi_\xi = 0 \tag{A.12}$$

We emphasize the fact that ϕ_ξ should be seen as a potential, such that a system living in this potential moves in the direction given by ξ. Therefore, a vector field ξ is hamiltonian if we can associate a potential to it.

We say that the action of G on M is hamiltonian if and only if, for every $g \in \mathfrak{g}$, ξ_g is hamiltonian. This means G acts on M hamiltonianely if and only if for any infinitesimal symmetry transformation of the space of configurations $g \in \mathfrak{g}$, we can associate a potential ϕ_g such that a system moving under this potential would move "in the direction of the symmetry".

We now come back to the Poisson structure on M. To any potential $H \in C^\infty(M)$, we associate the hamiltonian vector field ξ_H such that $\xi_H^b + \mathrm{d}H = 0$. This allows us to define the Poisson structure on M, by

$$\{F, G\} = \omega(\xi_F, \xi_G) \tag{A.13}$$

From this, we say the action of G on M is Poisson if we have for any two elements $g, h \in \mathfrak{g}$

$$\phi_{[g,h]} = \{\phi_g, \phi_h\} \tag{A.14}$$

meaning that the Poisson structure on M is compatible with the Lie structure on $\mathfrak{g}$.

Now, let's suppose that our theory is very regular, and that the action of G on M is Poisson. This allows us to freely speak of the potential ϕ_g associated to an element of $\mathfrak{g}$.

Remark. Recall however that the map

$$\begin{aligned} \mathfrak{g} &\to C^\infty(M) \\ g &\to \phi_g \end{aligned} \tag{A.15}$$

is very arbitrary. For $g \in \mathfrak{g}$, the function ϕ_g can be shifted arbitrarily by a function. This freedom will have its importance later on.

We define the *moment map* of the action by

$$\begin{aligned} \mu : M &\to \mathfrak{g}^* \\ \mu(p)(g) &= \phi_g(p) \end{aligned} \tag{A.16}$$

This map can be seen as a dual to the map $g \to \phi_g$. It takes a point in M, and associate to it some kind on potential on $\mathfrak{g}$. This association is very dependant on the choice of the potentials $(\phi_g)_g$. To better interpret it, let's suppose we have a point $p \in M$ such that for all $g \in \mathfrak{g}$, $\phi_g(p) = 0$. This point can be seen as an origin for the potentials, the starting point of 0 energy.

Remark. Notice that due to the freedom in the choice of the potentials $(\phi_g)_g$, given a point p, we can always redefine them such that for all $g \in \mathfrak{g}$, $\phi_g(p) = 0$.

Then, given another point $m \in M$, for $g \in \mathfrak{g}$, $\mu(m)(g)$ is equal to the difference in "energy" associated to g between the points p and m. Therefore, looking at $\mu(m)$ as an element of $\mathfrak{g}$, we understand that $\mu(m)$ points in the direction of the symmetries which have been used to go from p to m. We can see $\mu(m)$ as the momentum of the point m starting from p in the space of symmetries, which explains the terminology "moment map". In particular, we see that the set of points m for which $\mu(m) = 0$ is the set of point having the same "energy" as p according to all symmetries g. Looking at the symmetries of the space as gauge redundancies, we see that this set is the set of all points of M having the same gauge fixing.

A.3 Classical Hamiltonian reduction

We know have the tools to define an Hamiltonian reduction. Suppose our theory has a set of symmetries $H \subset G$ that we want to remove. We can see the symmetries H as generators for gauge redundancies. We will then have to fix the gauge. We first fix a point $p \in M$, in the gauge we want to keep. We therefore want to keep all of the points of M having the same gauge as p, and only those. We define the momentum map μ of H, where the potential fields have been fixed with p for origin (such that $\phi_g(p) = 0$ for all $g \in \mathfrak{g}$). From the discussion above, we understand that the set of points we want to keep is

$$M_0 \equiv \{m \in M/\mu(m) = 0\} \tag{A.17}$$

If 0 is a regular value of μ, then M_0 is a closed embedded submanifold of M. Thanks to the fact that the action of H is Poisson, the momentum map is H-equivariant for the coadjoint representation on $\mathfrak{g}$, which implies that H preserves M_0. We have gauged out the redundancy. We still need to quotient out the symmetry. This can be simply done by considering $\tilde{M} \equiv M_0/H$, which is a smooth manifold if the H-action is free and proper. A central theorem in sympleptic geometry is then that if all of the conditions aforementioned are met, then $\tilde{M}$ can be equipped with a unique sympleptic structure, induced by the sympleptic structure on M. $\tilde{M}$ is called the Hamiltonian reduction of M. Its construction can be resumed by the following diagram

$$\begin{array}{ccc}
\mu^{-1}(\{0\}) & \hookrightarrow & M \\
\downarrow & & \downarrow \\
\tilde{M} & \hookrightarrow & M/H
\end{array}$$

Up until now, our discussion has been purely geometric. We should now discuss of an equivalent reduction, but from an algebraic point of view.

So far, we have discussed of manifolds, which are by nature geometric. What is the algebraic object associated to a manifold? We may study the space of smooth functions on manifolds instead, which has an algebraic structure and which contains most of the informations on the manifold itself. We would therefore like to construct $C^\infty(\tilde{M})$ from $C^\infty(M)$ and from the action of H on M.

The first step would be to construct $C^\infty(M_0)$. We have $M_0 = \mu^{-1}(\{0\})$, so we want to construct functions who only matter where μ is equal to 0. We start by the opposite, and define the vanishing ideal of M_0

$$\mathscr{I} = \{f \in C^\infty(M) \ \ / \ \ \forall p \in M, \ \mu(p) = 0 \Rightarrow f(p) = 0\} \tag{A.18}$$

$\mathscr{I}$ corresponds to the Poisson ideal where M_0 doesn't matter. Then, we naturally have

$$C^\infty(M_0) = C^\infty(M)/\mathscr{I} \tag{A.19}$$

Now, we want to express $C^\infty(\tilde{M})$, which is the set of functions of $C^\infty(M_0)$ which are constant on the leaves of the action of H. But the tangent of these leaves are precisely the hamiltonian vector fields whose functions are in $\mathscr{I}$, such that

$$C^\infty(\tilde{M}) = \{f \in C^\infty(M_0) \ / \ \{f, \mathscr{I}\} = 0\} \tag{A.20}$$

Going back to the full space M, this writes

$$C^\infty(\tilde{M}) = \{f \in C^\infty(M_0) \ / \ \{f, \mathscr{I}\} \subset \mathscr{I}\} \ / \ \mathscr{I} \tag{A.21}$$

We recognize here the Poisson normalizer $N(\mathscr{I})$ of $\mathscr{I}$ in $C^\infty(M)$, which allows us to finally write

$$C^\infty(\tilde{M}) = N(\mathscr{I}) \ / \ \mathscr{I} \tag{A.22}$$

Once again, this reduction can be pictured in the following diagram

$$\begin{array}{ccc} C^\infty(M)/\mathscr{I} & \hookrightarrow & C^\infty(M) \\ \big\downarrow & & \big\downarrow \\ C^\infty(\tilde{M}) & \hookrightarrow & N(\mathscr{I}) \end{array}$$

The advantage of the algebraic point of view is that it can easily be generalized to any kind of system, without necessarily any physical meaning. This also helps relax some conditions, allowing for a more extensive study of Hamiltonian reductions.

A.4 The BRST procedure

We have seen how to reduce a system by removing its symmetries, geometrically and algebraically. The objective of such a reduction, especially algebraically, is to afterwards be able to compute things from it. However, the structure of the reducted algebra $C^\infty(\tilde{M})$ might not always be clear. In particular, it may be difficult to find and compute $\mathscr{I}$, which makes it difficult algebraically to study $C^\infty(\tilde{M})$. This motivates the use of a better technique, which should give some algebra isomorphic to $C^\infty(\tilde{M})$ but constructed in a much more clear and straightforward way, without using $\mathscr{I}$.

We once again consider a physical system whose space of configuration is given by M, on which a symmetry group H acts, and which we want to reduce using the moment map μ. We have seen that we can construct M_0 using the momentum map, such that the Hamiltonian reduced space is nothing more than the set of function on M_0 which are constant on the leaves of the action of H, or equivalently which are invariant under the action of $\mathfrak{h}$ assuming H is connected.

$$C^\infty(\tilde{M}) = C^\infty(M_0)^{\mathfrak{h}} \tag{A.23}$$

This can be constructed using the cohomology of $\mathfrak{h}$ on $C^\infty(M_0)$, from the Chevalley-Eilenberg complex. We now proceed to this construction.

We want to find the elements of $\mathcal{M} \equiv C^\infty(M_0)$ invariant under the action of $\mathfrak{h}$. To do so, let's try to register the action of $\mathfrak{h}$ on $\mathcal{M}$. We first define the space of p-forms on $\mathfrak{h}$ with values in $\mathcal{M}$

$$C^p(\mathfrak{h}, \mathcal{M}) \equiv \mathrm{Hom}(\Lambda^p \mathfrak{h}, \mathcal{M}) \simeq \Lambda^p \mathfrak{h}^* \otimes \mathcal{M} \tag{A.24}$$

We equip these spaces with a derivative $d : C^p(\mathfrak{h}, \mathcal{M}) \to C^{p+1}(\mathfrak{h}, \mathcal{M})$ defined by

- $(\mathrm{d}m)(h) = h \cdot m \quad \text{for } m \in \mathcal{M}, h \in \mathfrak{h}$
- $(\mathrm{d}\alpha)(h, g) = -\alpha([h, g]) \quad \text{for } \alpha \in \mathfrak{h}^*, h, g \in \mathfrak{h}$
- $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \mathrm{d}\beta$
- $\mathrm{d}(\omega \otimes m) = \mathrm{d}\omega \otimes m + (-1)^{|\omega|} \omega \wedge \mathrm{d}m$

which results in the complex

$$\xrightarrow{\mathrm{d}} C^{p-1}(\mathfrak{h}, \mathcal{M}) \xrightarrow{\mathrm{d}} C^p(\mathfrak{h}, \mathcal{M}) \xrightarrow{\mathrm{d}} C^{p+1}(\mathfrak{h}, \mathcal{M}) \xrightarrow{\mathrm{d}}$$

This complex forms a very powerful tool, called the Chevalley-Eilenberg complex, which registers a lot of the structure of the action of $\mathfrak{h}$. Considering the natural action of $\mathfrak{h}$ onto itself, it is for exemple known that $H^1(\mathfrak{h}) = H^2(\mathfrak{h}) = 0$ if $\mathfrak{h}$ is semisimple. We can also easily see from the expression of the derivative that

$$H^0(\mathfrak{h}, \mathcal{M}) = \mathcal{M}^{\mathfrak{h}} \tag{A.25}$$

which is exactly what we wanted. We have

$$C^\infty(\tilde{M}) = H^0(\mathfrak{h}, C^\infty(M_0)) \tag{A.26}$$

However, this expression still depends on M_0, which we do not want. We will therefore try to first derive $C^\infty(M_0)$.

To do so, recall that $C^\infty(M_0) = C^\infty(M)/\mathscr{I}$. Therefore, if we manage to generate $\mathscr{I}$, we should be able to get $C^\infty(M_0)$ from an homology.

The key point is then to see that the ideal $\mathscr{I}$ corresponding to functions vanishing on M_0, that is on $\mu^{-1}(\{0\})$, is generated by the potentials (ϕ_h), which can be seen as the components of the map $\phi : h \to \phi_h$. This can be proved locally, with dimensionality arguments. Using this, we can create the 2-steps complex

$$\mathfrak{h} \otimes C^\infty(M) \xrightarrow{\quad\delta\quad} C^\infty(M) \xrightarrow{\quad\delta\quad} 0$$

where for $f \in C^\infty(M)$ we have $\delta f = 0$, and given a basis $(h_i)_i$ of $\mathfrak{h}$ for $\sum_i h_i \otimes f_i$ we have $\delta\left(\sum_i h_i \otimes f_i\right) = \sum_i f_i \phi_{h_i}$. In 0 degree, we then have that the homology of this complex is equal to

$$H^0_\delta = C^\infty(M)/\mathscr{I} = C^\infty(M_0) \tag{A.27}$$

However, this complex has non-trivial homology in positive degrees.

For i, we can rewrite f_i in term of a sum of $(\phi_{h_j})_j$, recalling $(\phi_{h_j})_j$ generates $\mathscr{I}$ and seeing that the part of f_i along M_0 will diseappear anyways since ϕ_i is equal to 0 on M_0. Writing

$$f_i = \sum_j f_{ij} \phi_{h_j} \tag{A.28}$$

we get

$$\delta\left(\sum_{ij} h_i \otimes f_{ij} \phi_{h_j}\right) = \sum_{ij} f_{ij} \phi_{h_i} \phi_{h_j} \tag{A.29}$$

which is equal to zero if $f_{ij} = -f_{ji}$. But these cocycles may be killed by extending the complex as

$$\Lambda^2\mathfrak{h} \otimes C^\infty(M) \xrightarrow{\quad\delta\quad} \mathfrak{h} \otimes C^\infty(M) \xrightarrow{\quad\delta\quad} C^\infty(M) \xrightarrow{\quad\delta\quad} 0$$

and by extending δ as an odd derivation

$$\delta(h \wedge g \otimes f) = h \otimes \phi_g f - g \otimes \phi_h f \tag{A.30}$$

Once again, we will get a non trivial homology at the left, which can be killed once again by extending the complex to the left. We can continue this process indefinitely, which lead to what is called the Koszul complex. Defining $K^q = \Lambda^q\mathfrak{h} \otimes C^\infty(M)$ and extending δ on these spaces as a derivation, the Koszul complex is given by $(K^\bullet, \delta)$:

$$\xrightarrow{\delta} K^{q+1} \xrightarrow{\delta} K^{q} \xrightarrow{\delta} K^{q-1} \xrightarrow{\delta}$$

Its homology is 0 for all degrees higher than 0, and we have

$$H^0(K^\bullet, \delta) = C^\infty(M_0) \tag{A.31}$$

which is exactly what we want. We may finally augment this complex as into an exact sequence

$$\xrightarrow{\delta} K^1 \xrightarrow{\delta} K^0 \xrightarrow{\epsilon} C^\infty(M_0) \longrightarrow 0$$

where ϵ is the canonical projection. Thisis called the Koszul resolution.

Now, in order to derive $C^\infty(\tilde{M})$, one can take the homology of the Koszul complex, and then take the cohomology of its Chevalley-Eileinberg complex. However, one strong result of homology is that we may do both at the same time, for a more straightforward result. The resulting complex will be called the BRST complex.

We want to take both the homology and the cohomology at the same time. We therefore consider the complex

$$C^{p,q} = C^p(\mathfrak{h}; K^q) = \Lambda^p\mathfrak{h}^* \otimes \Lambda^q\mathfrak{h} \otimes C^\infty(M) \tag{A.32}$$

This complex can be equipped with the two derivatives d and δ, where δ and ϵ simply ignore the components in $\Lambda^p\mathfrak{h}^*$, whilst d acts as the derivative of the Chevalley-Eileinberg complex of $\mathfrak{h}$ with values in the $\mathfrak{h}$-module $\Lambda^q\mathfrak{h} \otimes M$. Moreover, the two derivatives commute, meaning the following diagram is commutative

$$\begin{array}{ccc} C^{p,q} & \xrightarrow{\delta} & C^{p,q-1} \\ \downarrow{\scriptstyle \mathrm{d}} & & \downarrow{\scriptstyle \mathrm{d}} \\ C^{p+1,q} & \xrightarrow[\delta]{} & C^{p+1,q-1} \end{array}$$

Since the homology of the Koszul complex is concentrated in 0 degree, we can hope to only get the Lie algebra cohomology of $\mathfrak{h}$ with values in $C^\infty(M_0)$ by taking a derivative like $D = \delta + \mathrm{d}$, as to finally get $C^\infty(\tilde{M})$ in 0 degree.

$D = \delta + \mathrm{d}$ is a good candidate for a derivative on this extended complex. However, a derivative needs to be closed, and δ and d commute. As a consequence, the derivative isn't closed. To correct this, we would like for d and δ to anticommute. This can be done by adding an alternating sign in front of δ, by defining $\delta_1 = (-1)^p\delta$ on $C^{p,q}$. Writing $D = \mathrm{d} + \delta_1$, we then have $D^2 = 0$ as it should be.

D goes from $C^{q,p}$ to $C^{p+1,q} \oplus C^{p,q-1}$. This motivates the notion of total degree $n = p - q$. Defining the space of total degree n as

$$\mathscr{C}^n = \bigoplus_{p-q=n} C^{p,q} \tag{A.33}$$

we have that D goes from $\mathscr{C}^n$ to $\mathscr{C}^{n+1}$. We call $(\mathscr{C}^\bullet, D)$ the total complex. Now, the main result is that its cohomology is equal to $H(\mathfrak{h};\ C^\infty(M_0))$, namely the Lie algebra cohomology of $\mathfrak{h}$ with values in $C^\infty(M_0)$. In particular, we have

$$H^0(\mathscr{C}^\bullet) = C^\infty(\tilde{M}) \tag{A.34}$$

Details of the proof can be found in [Fig06].

A.5 Ghosts and antighosts

We have derived a method to get $C^\infty(\tilde{M})$ in a way that is computable. But how can it be used in physics, and what is its interpretation? To answer this, we should reformulate the BRST procedure in the same langage as the original theory.

To do so, the first step is to see that the total space $\mathscr{C}^\bullet$ has a physical structure. We have seen in A.1 that for a given space of configurations M, $C^\infty(M)$ should be equipped with a Poisson algebra structure. We should therefore expect that $\mathscr{C}^\bullet$ can also be equipped with a similar structure, giving it physical meaning. In fact, the right structure to give to $\mathscr{C}^\bullet$ is that of a (graded) Poisson superalgebra.

The space $C^\infty(M)$ naturally has a Poisson superalgebra structure without odd part. Moreover, we can give $\Lambda(\mathfrak{h} \oplus \mathfrak{h}^*)$ a Poisson superalgebra structure, by defining the product as the wedge product, by defining the bracket for $X, Y \in \mathfrak{h}$ and $\alpha, \beta \in \mathfrak{h}^*$ as

$$\begin{aligned} \{\alpha, X\} &= \{X, \alpha\} = \alpha(X) \\ \{X, Y\} &= \{\alpha, \beta\} = 0 \end{aligned} \tag{A.35}$$

and by extending it as an odd derivation. We then only need to combine the 2 structures together, which can be done by setting for $a, b \in \Lambda(\mathfrak{h} \oplus \mathfrak{h}^*)$, $f, g \in C^\infty(M)$,

$$\begin{aligned} (a, f)(b, g) &= (-1)^{|f||b|}(ab, fg) \\ \{(a, f),\ (b, g)\} &= (-1)^{|f||b|}\left((\{a, b\}, fg) + (ab, \{f, g\})\right) \end{aligned} \tag{A.36}$$

Therefore, $\mathscr{C}^\bullet$ has a Poisson superalgebra structure. We can also give it a $\mathbb{Z}$-grading, using the total degree defined earlier. D should then be a Poisson derivation of degree 1. We can then hope for D to be an inner Poisson derivation, meaning we can hope for the existence of an element Q of the total space $\mathscr{C}^\bullet$ such that $\{Q, -\} = D$. This would give a nice expression of D inside of the theory's framework, and would allow for physical interpretations.

Let's therefore search for such an element Q. We set $(X_i)_i$ a basis for $\mathfrak{h}$, and $(\alpha^i)_i$ its dual basis on $\mathfrak{h}^*$. We also set f_{ij}^k the structure constants of $\mathfrak{h}$, such that $[X_i, X_j] = f_{ij}^k X_k$. Since D and $\{Q, -\}$ are derivations, it is enough to find an element Q which acts well on the generators of $\mathscr{C}^\bullet$, meaning on functions $g \in C^\infty(M)$, on elements $Y \in \mathfrak{h}$ and on elements $\beta \in \mathfrak{h}^*$. We write $Y = Y^i X_i$ and $\beta = \beta_i \alpha^i$ the decompositions of Y and β in their respective basis.

On these generators, we know that δ ignores β, that $\delta f = 0$ and that $\delta Y = Y^i \phi_{X_i}$. Moreover, we know that

$$\begin{aligned}
\mathrm{d}f(Z) &= \{\phi_Z, f\} = Z^i\{\phi_{X_i}, f\} \\
\mathrm{d}Y(Z) &= [Z, Y] = f_{jk}^i Z^j Y^k X_i \\
\mathrm{d}\beta(Z, Z') &= -\beta([Z, Z']) = -\frac{1}{2}\beta_i Z^j Z^k f_{jk}^i
\end{aligned} \tag{A.37}$$

such that

$$\begin{aligned}
\mathrm{d}f &= \alpha^i\{\phi_{X_i}, f\} \\
\mathrm{d}Y &= f_{jk}^i Y^k \alpha^j \wedge X_i \\
\mathrm{d}\beta &= -\frac{1}{2}\beta_i f_{jk}^i \alpha^j \wedge \alpha^k
\end{aligned} \tag{A.38}$$

Summing the two derivatives, we should therefore have

$$\begin{aligned}
\{Q, f\} &= \alpha^i\{\phi_{X_i}, f\} \\
\{Q, Y\} &= Y^i \phi_{X_i} + f_{jk}^i Y^k \alpha^j \wedge X_i \\
\{Q, \beta\} &= -\frac{1}{2}\beta_i f_{jk}^i \alpha^j \wedge \alpha^k
\end{aligned} \tag{A.39}$$

We may then see that taking

$$Q = \alpha^i \phi_{X_i} - \frac{1}{2} f_{jk}^i \alpha^j \wedge \alpha^k \wedge X_i \tag{A.40}$$

gives exactly the equations above. This proves that D is an inner derivation, with inner element Q. Q is usually called the BRST operator. We may rewrite X_i as b_i, α^i as c^i, ϕ_{X_i} as ϕ_i, and drop the mention to the wedge product, as to get the convenient expression

$$Q = c^i \phi_i - \frac{1}{2} f_{jk}^i c^j c^k b_i \tag{A.41}$$

which is the most common expression in physics literature.

How does one interpret this? Let's first look at the expanded space. The space of functions on the space of configurations $C^\infty M$ has been extended to the sum

$$\mathscr{C}^\bullet = \bigoplus_{p,q} C^{p,q} = \bigoplus_{p,q} \Lambda^p \mathfrak{h}^* \otimes \Lambda^q \mathfrak{h} \otimes C^\infty(M) \tag{A.42}$$

Looking at each $C^{p,q} = \Lambda^p \mathfrak{h}^* \otimes \Lambda^q \mathfrak{h} \otimes C^\infty(M)$ individually, we see that they can be interpreted as

$$C^{p,q} \simeq C^\infty\left(M \sqcup \left(\bigsqcup_{k=1}^{p} \{c^1, \ldots, c^r\}_k\right) \sqcup \left(\bigsqcup_{k=1}^{q} \{b_1, \ldots, b_r\}_k\right)\right) \tag{A.43}$$

Recalling that M represents the space of configurations, we can understand $C^{p,q}$ as the space describing a system whose configuration can be either in M as before, either an element of the basis of one of the p different copies of $\mathfrak{h}^*$, either an element of the basis of one of the q different copies of $\mathfrak{h}$. Then, $\mathscr{C}^\bullet$ is the space of function on the disjoint sum of all of these spaces, and can be seen as the space describing a system whose configuration is described by one of the $C^{p,q}$.

This classical description can seem very messy. However, moving our theory to the QFT framework, we can get a much nicer intuition. Let's suppose our theory is a QFT. M is then the space of configuration of a "particle" in the theory. But in a QFT, the exact state of a system and exact number of "particles" are not known, such that an actual state of the system (a field φ) is a distribution on M, meaning $\varphi \in C^\infty(M)$[2]. The height of φ at some point $x \in M$ is then the probability for this system to have particles in the configuration x.

From this perspective, upgrading $C^\infty(M)$ to $\mathscr{C}^\bullet$ means upgrading fields from being elements of $C^\infty(M)$ to being elements of $\mathscr{C}^\bullet$. A given field is then a sum of fields in the $(C^{p,q})_{p,q}$, meaning a given state of the system is a sum of particles in the $(C^{p,q})_{p,q}$. But given p, q, a field in $C^{p,q}$ is a field on M added to fields on p copies of $\{c^1, \dots, c^r\}$ and q copies of $\{b_1, \dots, b_r\}$. Furthermore, a field on $\{c^1, \dots, c^r\}$ simply represents a set of particles which can be in the configurations $c^1, \dots, c^r$, whilst a field on $\{b_1, \dots, b_r\}$ simply represents a set of particles which can be in the configurations $b_1, \dots, b_r$.

We call *ghosts* the particles in one of the configurations $c^1, \dots, c^r$, whilst we call *antighosts* the particles in one of the configurations $b_1, \dots, b_r$. We also call ghost fields and antighost fields the associated fields. A ghost particle can be imagined virtually as a direction in the Lie group H: if the particle is determined, it is necessarily one of the generators of the Lie algebra. The same thing is valid for antighosts. With this intuition, a field on $C^{p,q}$ is nothing more than particles as described by the original theory, with the addition of an arbitrary number of ghosts and antighosts, where we can distinguish p different types of ghosts and q different types of antighosts. The complete theory then describe an arbitrary number of particles as desrcibed by the original theory, but where they are each coupled to an arbitrary number of ghosts and antighosts, with each time an arbitrary number of different possible kinds of ghosts and antighosts.

Ghosts and antighosts are understandably called this way, because they are particles which only interact with particles from the original theory. Therefore, the only way to probe them is indirectly, by measuring their effects on the particles from the original theory. Why do new particles appear when trying to reduce the theory? We can think of the ghosts and antighosts as particles registering the geometry of H. Then, to impose constraints on the original particles taking into account the geometry of H, we must consider a new extended theory where there are also ghosts and antighosts, as to allow for interactions between ghosts, antighosts, and the original particles, using the ghosts and antighosts to impose the constraints we want on the original particles. Ghosts and antighosts

[2] Here, we do not make the difference between distributions and smooth functions, as we are only interested in intuition

can also be thought of as walls which prevent the original particles from going into directions we do not want.

Remark. From the point of view of a mathematician, it might seem weird to interpret the ghosts and antighosts as particles. However, historically, this is how they were first introduced.

Now, how does one intepret the BRST operator Q? Q is a field of the theory, which can be seen as a function on the space of configurations as described by (A.43). But from the point of view of Hamiltonian mechanics described in A.1, we can see this function as a potential, giving some kind of potential energy to each configuration of the system. Then, for another given field φ of the theory, its time evolution in this energy potential is given by $\{\varphi, Q\}$. In particular, if $\{\varphi, Q\} = 0$, it means that φ is stationnary in the potential given by φ, meaning it doesn't change.

With this point of view, let's look at what actually is the reduced space. The reduced space is the cohomology of the complete space $\mathscr{C}^\bullet$ by the inner derivative generated by Q, in degree 0. First, the condition to keep only the degree 0 of the cohomology means in any state of the system, there should always be one more type of ghost than antighost. Seeing ghosts as walls and antighosts as wall, this is similar to saying that walls should always be of dimension 1. Note that in some cases, we will have that the cohomology of the BRST complex is concentrated in degree 0 anyways, meaning we can forget about this condition.

Now, what does taking the cohomology really mean? We are taking the kernel of the derivative, and quotienting out its image. This means that we are only considering fields φ such that $\{\varphi, Q\} = 0$, meaning we only consider fields that do not change under the potential Q. Seeing $\{Q, -\}$ as the generator for the symmetries we want to quotient out, this is expected since we are then asking for φ to be unchanged under the action of the symmetry. Moreover, we do not consider fields φ if there exists another field ϕ in the complete space such that $\{\phi, Q\} = \varphi$, meaning we do not consider fields that give the flow of another field in the potential energy given by Q. Looking back at Q, we see that it describes a potential energy linked to the ones given by the momentum map, but with a ghost associated to each potential field given by the momentum map. It also has a component in the space with 2 ghosts and 1 antighost, as some sort of momentum on the ghost fields. Seeing this, we may understand how taking the fields stationnary under this energy can yields the same result as the classical Hamiltonian reduction seen in A.3.

Appendix B

Good gradings

This appendix serves as a quick discussion of the idea of good gradings, and aims at giving a few equivalent definitions to this notion. We will not prove most results, and refer to [EK04] for detailed proofs and a further discussion on good gradings of Lie algebras.

The adjective "good" of "good gradings" describes the G-grading of finite-dimensional Lie algebras, where G is a discrete additive subgroup of $\mathbb{R}$, that is of the form $a\mathbb{Z}$. Without loss of generality, we can choose any a and study only $a\mathbb{Z}$-gradings, as all such groups are isomorphics. [EK04] chooses for exemple $G = \mathbb{Z}$. However, we will take the more physical $G = \frac{1}{2}\mathbb{Z}$, which will also be the case we encounter in these notes.

B.1 Definition

We consider a $\frac{1}{2}\mathbb{Z}$-graded finite-dimensional Lie algebra $\mathfrak{g}$, that is a Lie algebra $\mathfrak{g}$ endowed with a decomposition in subspaces

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j \tag{B.1}$$

such that the decomposition is compatible with the Lie bracket, meaning

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \tag{B.2}$$

Remark. From a physical perspective where $\mathfrak{g}$ corresponds to the algebra of operators of a quantum field theory, j can be seen as the spin of the fields generated by the operators in $\mathfrak{g}_j$. The algebra of bosonic operators is then $\mathfrak{g}_b = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$, whilst the algebra of fermionic operators is $\mathfrak{g}_f = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_{j+\frac{1}{2}}$.

We define according to the usual conventions

$$\begin{aligned} \mathfrak{g}_+ &= \bigoplus_{j>0} \mathfrak{g}_j & \mathfrak{g}_- &= \bigoplus_{j<0} \mathfrak{g}_j \\ \mathfrak{g}_\geq &= \bigoplus_{j\geq 0} \mathfrak{g}_j & \mathfrak{g}_\leq &= \bigoplus_{j\leq 0} \mathfrak{g}_j \end{aligned} \tag{B.3}$$

Let e an element of $\mathfrak{g}_1$. Due to the compatibility of the grading with the Lie bracket, the adjoint action of $\mathfrak{g}$ gives the map $\mathrm{ad}(e)$ which to any element of $\mathfrak{g}_j$ associates an element of $\mathfrak{g}_{j+1}$. Therefore, $\mathrm{ad}(e)$ creates two disjoint chains on $\mathfrak{g}$, one on $\mathfrak{g}_b$ and the other on $\mathfrak{g}_f$.

We say that e is *good* if

$$\mathrm{ad}(e) : \mathfrak{g}_j \to \mathfrak{g}_{j+1} \quad \text{is injective for } j < 0 \tag{B.4}$$

$$\mathrm{ad}(e) : \mathfrak{g}_j \to \mathfrak{g}_{j+1} \quad \text{is surjective for } j \geq -\frac{1}{2} \tag{B.5}$$

This implies in particular that

$$\mathrm{ad}(e) : \mathfrak{g}_{-\frac{1}{2}} \to \mathfrak{g}_{\frac{1}{2}} \quad \text{is an isomorphism} \tag{B.6}$$

Note that (B.4) is equivalent to

$$\mathfrak{g}^e \subset \mathfrak{g}_{\geq} \tag{B.7}$$

We say that a grading is *good* when there exist an element $e \in \mathfrak{g}_1$ such that e is good.

An important class of such good gradings is given by triples $\{e, h, f\}$ such that $[e, f] = h, [h, e] = e, [h, f] = -f$. The grading given by the eigenspace decomposition of h is a good $\frac{1}{2}\mathbb{Z}$-grading, with good element e. Notice that if we had chosen to study $\mathbb{Z}$-gradings instead of $\frac{1}{2}\mathbb{Z}$-gradings, we would have the same result for $\{e, h, f\}$ an sl_2-triple. The result is then well-known from the representation theory of sl_2.

B.2 Good grading of semisimple Lie algebras

We would like to see how much the above exemple is "classical", and how far can a good grading stray from it. We suppose now and for the remaining of this appendix that $\mathfrak{g}$ is semisimple.

All derivations of $\mathfrak{g}$ are inner. But the "spin" map $\jmath : \mathfrak{g} \to \mathfrak{g}$ defined for all $j \in \mathbb{Z}$, $x \in \mathfrak{g}_j$ by $\jmath(x) = jx$ can easily be shown to verify the Leibnitz rule. So it is a derivation and there exist $x \in \mathfrak{g}$ such that $\jmath = \mathrm{ad}(x)$. The $(\mathfrak{g}_j)_j$ can be seen as the eigenspaces of $\mathrm{ad}(x)$, with eigenvalues j. We say that x defines the grading. Reciprocally, given any element $x \in \mathfrak{g}$, the eigenspace decomposition of its adjunction gives a grading on $\mathfrak{g}$, where the eigenvalues give a label on the decomposition subspaces.

Suppose now that the grading on $\mathfrak{g}$ is good, and let e a good element. Adapting the Jacobson Morozov theorem to our case, we can find $\tilde{h}, \tilde{f} \in \mathfrak{g}$ such that $[e, \tilde{f}] = \tilde{h}, [\tilde{h}, e] = e, [\tilde{h}, \tilde{f}] = -\tilde{f}$. But then, by projecting onto the subspaces given by the grading and by re-using the theorem, we can find $h \in \mathfrak{g}_0, f \in \mathfrak{g}_{-1}$ such that $[e, f] = h, [h, e] = e, [h, f] = -f$.

Theorem B.2.1 ([EK04], 1.1)**.** *Given $\mathfrak{g}$ a semisimple Lie algebra equipped with a good grading and a good element e, let $x, h, f \in \mathfrak{g}$ such that the grading is generated by x, and such that $[e, f] = h, [h, e] = e, [h, f] = -f$. Then*

$$x - h \quad \textit{lies in the center of} \quad \mathfrak{g}^{\{e,h,f\}} \tag{B.8}$$

In particular, if the center of $\mathfrak{g}^{\{e,h,f\}}$ is trivial, then $x = h$ and we are in the case of our exemple above.

$\mathfrak{g}_0$ is a reductive subalgebra of $\mathfrak{g}$, and a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_0$ is also a Cartan subalgebra of $\mathfrak{g}$. We let

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right) \tag{B.9}$$

the root space decomposition of $\mathfrak{g}$ according to $\mathfrak{h}$. Writing Δ_0^+ a system of positive roots of the subalgebra $\mathfrak{g}_0$, the set $\Delta^+ = \Delta_0^+ \cup \{\alpha \in \Delta \ / \ \mathfrak{g}_\alpha \subset \mathfrak{g}_+\}$ is a system of positive roots. Let then $\Pi \subset \Delta^+$ the set of simple roots. We decompose Π according to the grading, setting $\Pi_j = \{\alpha \in \Pi \ / \ \mathfrak{g}_\alpha \subset \mathfrak{g}_j\}$. We then have the following important result:

Theorem B.2.2 ([EK04], 1.2)**.** *If $\mathfrak{g}$ is a semisimple Lie algebra equipped with a good grading, then $\Pi = \Pi_0 \cup \Pi_{\frac{1}{2}} \cup \Pi_1$, with Π and Π_j defined as above.*

Another result which follows is

Theorem B.2.3 ([EK04], 1.3)**.** *The properties* (B.4) *and* (B.5) *are equivalent in the definition of a good element of a semisimple Lie algebra*

As such, given a semisimple Lie algebra $\mathfrak{g}$, let a pair of elements (x, e) (resp. (x, f)) such that x is ad-diagonalizable with half-integer eigenvalues, such that $[x, e] = e$ (resp. $[x, f] = -f$) and such that the centralizer of e (resp. f) lies in $\mathfrak{g}_{\geq}$ (resp. $\mathfrak{g}_{\leq}$) with the notations of (B.3) where the grading structure is given by the eigenspace decomposition of x. Then the pair gives a structure of good grading on $\mathfrak{g}$ where the $\frac{1}{2}\mathbb{Z}$-grading is given by the eigenspace decomposition of x. In this case, we call (x, f) a *good pair*, following [DK06]. Reciprocally, any good grading on a semisimple Lie algebra can be described by a good pair.

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