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Supersymmetric *W*-algebras

Alexander Molev[1] · Eric Ragoucy[2] · Uhi Rinn Suh[3]

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Abstract
We explain a general theory of W-algebras in the context of supersymmetric vertex algebras. We describe the structure of W-algebras associated with odd nilpotent elements of Lie superalgebras in terms of their free generating sets. As an application, we produce explicit free generators of the W-algebra associated with the odd principal nilpotent element of the Lie superalgebra $\mathfrak{gl}(n+1|n)$.

Keywords Supersymmetric Vertex algebras · W superalgebras · Free generating sets

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1 Introduction

The W-algebras first appeared in relation with the conformal field theory in the work of Zamolodchikov [26] and Fateev and Lukyanov [11]. These algebras were studied intensively by physicists, both at the classical level through Hamiltonian reduction of Wess–Zumino–Novikov–Witten models and their connection with affine Lie algebras, see, e.g., [5,12,14], but also using BRST formalism [7,8]. For an extensive review on physicists works, see [6] and references therein. A definition of the W-algebras in the context of the vertex algebra theory and quantized Drinfeld–Sokolov reduction was

✉ Uhi Rinn Suh
uhrisu1@snu.ac.kr

Alexander Molev
alexander.molev@sydney.edu.au

Eric Ragoucy
eric.ragoucy@lapth.cnrs.fr

1 School of Mathematics and Statistics, University of Sydney, Sydney, NSW 2006, Australia

2 Laboratoire de Physique Théorique LAPTh, CNRS, Université Savoie Mont Blanc, BP 110, 74941 Annecy-le-Vieux Cedex, France

3 Department of Mathematical Sciences and Research institute of Mathematics, Seoul National University, GwanAkRo 1, Gwanak-Gu, Seoul 08826, Korea

given by Feigin and Frenkel [13]; see also the book by Frenkel and D. Ben-Zvi [15, Ch. 15]. A more general family of W-algebras $W^k(\mathfrak{g}, f)$ was introduced in the articles by Kac, Roan and Wakimoto [21] and Kac and Wakimoto [22], which depends on a simple Lie (super)algebra $\mathfrak{g}$, an (even) nilpotent element $f \in \mathfrak{g}$ and the *level* $k \in \mathbb{C}$. In the particular case of the principal nilpotent element $f = f_{\text{prin}}$, this reduces to the definition of [13]; see also a recent expository article by Arakawa [2] where basic structure theorems and representation theory of W-algebras are reviewed.

In the present paper, we will be concerned with supersymmetric counterparts of the W-algebras which can be defined by analogy with [15, Ch. 15]. Such W-algebras have already been studied, mostly in the physics literature; see [10,17,18]. In this context, they are viewed as a generalization of the 'usual' W-algebras to the case of superconformal theories. The main tool for their study is the superspace and superfield formalism. Moreover, a supersymmetric quantum Hamiltonian reduction approach was developed in the work of Madsen and the second author [23]. We will rely on this work and the supersymmetric vertex algebra theory developed by Heluani and Kac [16,19] to describe the structure of the W-algebras associated with odd nilpotent elements of Lie superalgebras. The supersymmetric vertex algebra theory can be viewed as the mathematical counterpart of the superspace formalism used in physics. Our main structural result is Theorem 4.11 which describes free generating sets of the W-algebras.

We will then apply the main result to the case of the general linear Lie superalgebras. It is well known that the Lie superalgebra $\mathfrak{gl}(m|n)$ contains an odd principal nilpotent element if and only if $m = n \pm 1$. We take $m = n + 1$ (this can be done without a real loss of generality) and produce explicit free generators of the W-algebra as coefficients of a certain non-commutative characteristic polynomial; see Theorems 5.1 and 5.3. These formulas can be regarded as supersymmetric analogues of the generators of the principal W-algebra associated with the Lie algebra $\mathfrak{gl}(n)$ (which can be identified with the W_n-algebra [11] via the Miura map), as produced by Arakawa and the first author [3]. In particular, when considering the Lie superalgebra $\mathfrak{gl}(2|1)$, one gets the superconformal (or super-Virasoro) algebra with 2 fermionic generators [1], [24]. Furthermore, we show that the Miura transformation used in [3] can also be applied in the supersymmetric context to recover the generators of the W-algebra appeared in [10,17,18].

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2 Supersymmetric vertex algebras

In this section, we introduce supersymmetric vertex algebras following [16] and [19]. Proofs and additional details can be found in these references. Note that in the terminology of the paper [16] these objects are called $N_K = 1$ *supersymmetric vertex algebras*.

2.1 Notation and basic definitions

We will be considering two couples of coordinates

$$Z = (z, \theta), \quad W = (w, \zeta),$$

where z and w are even and θ and ζ are odd. Introduce the notation

$$\mathbb{C}[\![Z]\!] := \mathbb{C}[\![z]\!] \otimes \mathbb{C}[\theta], \quad \mathbb{C}(\!(Z)\!) := \mathbb{C}(\!(z)\!) \otimes \mathbb{C}[\theta].$$

Since $\theta^2 = 0$, we have $\mathbb{C}[\theta] = \mathbb{C} \oplus \mathbb{C}\theta$. Similarly,

$$\mathbb{C}[Z, Z^{-1}] := \mathbb{C}[z, z^{-1}] \otimes \mathbb{C}[\theta], \quad \mathbb{C}[\![Z, Z^{-1}]\!] := \mathbb{C}[\![z, z^{-1}]\!] \otimes \mathbb{C}[\theta].$$

Furthermore, set

$$\begin{aligned} &Z - W := (z - w - \theta\zeta, \theta - \zeta), \\ &Z^{j_0|j_1} := z^{j_0}\theta^{j_1} \quad \text{for } j_0 \in \mathbb{Z}, \ j_1 = 0, 1, \\ &(Z - W)^{j_0|j_1} := (z - w - \theta\zeta)^{j_0}(\theta - \zeta)^{j_1}. \end{aligned}$$

Let $\mathcal{U} = \mathcal{U}_{\bar{0}} \oplus \mathcal{U}_{\bar{1}}$ be a $\mathbb{Z}/2\mathbb{Z}$-graded vector space which we will also call a *vector superspace*. Accordingly, elements $a \in \mathcal{U}_{\bar{0}}$ (resp. $a \in \mathcal{U}_{\bar{1}}$) are called *even* (resp. *odd*) with the parity $p(a) = \bar{0}$ (resp. $p(a) = \bar{1}$). The corresponding endomorphism algebra $\operatorname{End}\mathcal{U} = (\operatorname{End}\mathcal{U})_{\bar{0}} \oplus (\operatorname{End}\mathcal{U})_{\bar{1}}$ is a superalgebra, where

$$f \in (\operatorname{End}\mathcal{U})_{\bar{\imath}} \quad \Longleftrightarrow \quad f\big((\operatorname{End}\mathcal{U})_{\bar{\jmath}}\big) \subset (\operatorname{End}\mathcal{U})_{\bar{\imath}+\bar{\jmath}}$$

for any $\bar{\imath}, \bar{\jmath} \in \mathbb{Z}/2\mathbb{Z}$.

Any element of the vector superspace $\mathcal{U}[\![Z, Z^{-1}]\!] := \mathcal{U} \otimes \mathbb{C}[\![Z, Z^{-1}]\!]$ is called a *$\mathcal{U}$-valued formal distribution*. It has the form

$$a(Z) = \sum_{j_0 \in \mathbb{Z},\ j_1 = 0,1} Z^{j_0|j_1} a_{j_0|j_1} \in \mathcal{U}[\![Z, Z^{-1}]\!], \qquad a_{j_0|j_1} \in \mathcal{U}. \tag{2.1}$$

The *super-residue* of a formal distribution $a(Z)$ is defined by

$$\operatorname{res}_Z a(Z) := a_{-1|1} \in \mathcal{U}.$$

Since $\operatorname{res}_Z Z^{j_0|j_1} a(Z) = a_{-1-j_0|1-j_1}$, it is convenient to use the notation

$$a_{(j_0|j_1)} := \operatorname{res}_Z Z^{j_0|j_1} a(Z)$$

so that $a_{j_0|j_1} = a_{(-1-j_0|1-j_1)}$ and the distribution $a(Z)$ in (2.1) takes the form

$$a(Z) = \sum_{j_0 \in \mathbb{Z},\ j_1 = 0,1} Z^{-1-j_0|1-j_1} a_{(j_0|j_1)}.$$

An End $\mathcal{U}$-valued formal distribution $a(Z)$ is called a *superfield* if for any given $v \in \mathcal{U}$ there exists $N \in \mathbb{Z}_{\geq 0}$ such that

$$a_{(j_0|j_1)}v = 0 \quad \text{for all} \quad j_0 \geq N\,, \; j_1 = 0, 1.$$

Similarly, a *$\mathcal{U}$-valued formal distribution in two variables* is an element of the vector superspace $\mathcal{U}[\![Z, Z^{-1}, W, W^{-1}]\!]$:

$$a(Z, W) = \sum_{\substack{j_0, k_0 \in \mathbb{Z}, \\ j_1, k_1 = 0, 1}} Z^{j_0|j_1} W^{k_0|k_1} a_{j_0|j_1, k_0|k_1} \in \mathcal{U}[\![Z, Z^{-1}, W, W^{-1}]\!]$$

with $a_{j_0|j_1,k_0|k_1} \in \mathcal{U}$. A formal distribution $a(Z, W)$ is called *local* if

$$(z - w)^n a(Z, W) = 0$$

for some $n \in \mathbb{Z}_{\geq 0}$. We let the *formal δ-distribution* be defined by

$$\delta(Z, W) = (\theta - \zeta) \sum_{n \in \mathbb{Z}} z^n w^{-n-1}.$$

Note that for any $f \in \mathcal{U}[\![Z, Z^{-1}]\!]$ we have

$$\text{res}_Z \delta(Z, W) f(Z) = f(W).$$

Since $(z - w)\delta(Z, W) = 0$, the formal δ-distribution is local.

The differential operators $\partial_z, \partial_\theta, \partial_w$ and ∂_ζ act naturally on $\mathbb{C}[\![Z, Z^{-1}, W, W^{-1}]\!]$. Consider two more odd differential operators

$$D_Z = \partial_\theta + \theta \partial_z, \quad D_W = \partial_\zeta + \zeta \partial_w.$$

Then, $[D_Z, D_Z] = 2\partial_z$. Set

$$D_Z^{j_0|j_1} = \partial_z^{j_0} D_Z^{j_1}, \quad D_Z^{(j_0|j_1)} = (-1)^{j_1} \frac{1}{j_0!} D_Z^{j_0|j_1}.$$

Lemma 2.1 *Let $a(Z, W)$ be a local formal distribution. Then,*

$$a(Z, W) = \sum_{\substack{j_0 \in \mathbb{Z}_{\geq 0}, \\ j_1 = 0, 1}} D_W^{(j_0|j_1)} \delta(Z, W)\, c_{j_0|j_1}(W),$$

where the sum is finite, and

$$c_{j_0|j_1}(W) = res_Z (Z - W)^{j_0|j_1} a(Z, W).$$

Definition 2.2 A *supersymmetric vertex algebra* is a tuple $(V, |0\rangle, S, Y)$ where V is a vector superspace, $|0\rangle \in V$ is a *vacuum vector*, S is an odd endomorphism of V, and the *state-field correspondence* Y is a parity preserving linear map from V to the space of End V-valued superfields

$$Y : V \to \operatorname{End} V[\![Z, Z^{-1}]\!], \quad a \mapsto a(Z)$$

satisfying the following axioms:

- (*vacuum*) $a(Z)\,|0\rangle\,|_{z=0,\,\theta=0} = a,\ S\,|0\rangle = 0$,
- (*translation covariance*) $[S, a(Z)] = (\partial_\theta - \theta\partial_z)a(Z)$,
- (*locality*) for any $a, b \in V$ there exists $N \in \mathbb{Z}_+$ such that $(z-w)^N[a(Z), b(W)] = 0$.

By Lemma 2.1, the locality axiom implies a finite sum decomposition

$$[a(Z), b(W)] = \sum_{\substack{j_0\in\mathbb{Z}_{\geq 0},\\ j_1=0,1}} \left(D_W^{(j_0|j_1)}\delta(Z, W)\right) a(W)_{(j_0|j_1)}b(W)$$

for $a(W)_{(j_0|j_1)}b(W) := \operatorname{res}_Z(Z - W)^{j_0|j_1}[a(Z), b(W)]$. The expression $a(W)_{(j_0|j_1)}$ $b(W)$ is called the $(j_0|j_1)$*-th product of the superfields* $a(W)$ *and* $b(W)$.

Definition 2.3 (1) The *normally ordered product* of two End V-valued formal distributions $a(Z)$ and $b(Z)$ is defined by

$$: a(Z)b(Z) := a_+(Z)b(Z) + (-1)^{p(a)p(b)}b(Z)a_-(Z),$$

where

$$a_+(Z) = \sum_{j_0\in\mathbb{Z}_{\geq 0},\ j_1=0,1} Z^{j_0|j_1}a_{j_0|j_1} \quad \text{and} \quad a_-(Z) = \sum_{j_0\in\mathbb{Z}_{<0},\ j_1=0,1} Z^{j_0|j_1}a_{j_0|j_1}.$$

(2) If $j_0 \leq -2$ and $j_1 = 0, 1$, or $j_0 = -1$ and $j_1 = 0$, then $a(Z)_{(j_0|j_1)}b(Z)$ is given by

$$a(Z)_{(j_0|j_1)}b(Z) = (-1)^{1-j_1} : \left(D_Z^{(-1-j_0|1-j_1)}a(Z)\right)b(Z) : .$$

Remark 2.4 One can check that

$$: a(Z)b(Z) : |0\rangle\,|_{z=0,\theta=0} = a_{(-1|1)}b$$

and

$$a(Z)_{(j_0|j_1)}b(Z)\,|0\rangle\,|_{z=0,\theta=0} = a_{(j_0|j_1)}b$$

for (j_0, j_1) as in part (2) of Definition 2.3.

Lemma 2.5 *(Dong's lemma) Let $a(Z), b(Z), c(Z)$ be pairwise local formal distributions. Then, $\big(a(Z), (b_{(j_0|j_1)}c)(Z)\big)$ is local for any $j_0 \in \mathbb{Z}$ and $j_1 = 0, 1$.*

Lemma 2.6 *(Uniqueness lemma) Let V be a supersymmetric vertex algebra. If $a(Z)$ is a superfield such that $(a(Z), b(Z))$ is local for every $b \in V$ and $a(Z)\,|0\rangle = 0$, then $a(Z) = 0$.*

By the uniqueness lemma and Remark 2.4,

$$a(Z)_{(j_0|j_1)}b(Z) = (a_{(j_0|j_1)}b)(Z),$$

and we set

$$: ab := a_{(-1|1)}b =: a(Z)b(Z) : |0\rangle\, |_{z=0,\,\theta=0}.$$

Note that for a given supersymmetric vertex algebra V, the state-field correspondence map

$$Y : V \to (\operatorname{End} V)[\![Z, Z^{-1}]\!], \quad a \mapsto a(Z),$$

is injective. Hence, a supersymmetric vertex algebra V can be considered as a set of superfields $Y(V)$. In the following theorem, we construct a vertex algebra as a set of superfields.

Theorem 2.7 *(Existence theorem) Let V be a vector superspace and $\widehat{V}$ be a set of pairwise local* End V*-valued superfields. Suppose $Id \in \widehat{V}$ is the constant field and $\widehat{V}$ is invariant under the operator $D = \partial_\theta + \theta\partial_z$ and all $(j_0|j_1)$-products. Then, the superspace V with the vacuum vector Id, the operator S given by $Sa(Z) = D(a(Z))$ and the $(j_0|j_1)$-products are a supersymmetric vertex algebra.*

2.2 Supersymmetric Lie conformal algebras

Recall that a Lie conformal algebra (LCA) R gives rise to a vertex algebra called a universal enveloping vertex algebra $V(R)$ [4,19]. Now we introduce its supersymmetric analogue: that is, a supersymmetric LCA and the corresponding universal enveloping supersymmetric vertex algebra. Consider two superalgebras:

- Let $\mathcal{L}$ be the associative superalgebra generated by a pair of elements $\Lambda = (\lambda, \chi)$, where λ is even and χ is odd, such that

$$[\lambda, \chi] = 0, \quad [\chi, \chi] = 2\chi^2 = -2\lambda.$$

- Let $\mathcal{K}$ be another associative superalgebra generated by a pair of elements $\nabla = (T, S)$, where T is even and S is odd, such that

$$[T, S] = 0, \quad [S, S] = 2S^2 = 2T.$$

Note that $\mathcal{L}$ and $\mathcal{K}$ are isomorphic via the map $\lambda \mapsto -T$ and $\chi \mapsto -S$.

Set

$$(Z-W)\Lambda = (z-w-\theta\zeta)\lambda + (\theta-\zeta)\chi.$$

Given a formal distribution $a(Z,W)$ of two variables Z and W, consider the *formal Fourier transformation*

$$\mathcal{F}^{\Lambda}_{Z,W}\, a(Z,W) = \mathrm{res}_Z \exp\big((Z-W)\Lambda\big) a(Z,W)$$

which can be expanded as

$$\mathcal{F}^{\Lambda}_{Z,W}\, a(Z,W) = \sum_{j_0\in\mathbb{Z}_{\geq 0},\, j_1=0,1} (-1)^{j_1} \Lambda^{(j_0|j_1)} c_{j_0|j_1}(W),$$

where

$$\Lambda^{(j_0|j_1)} = (-1)^{j_1} \frac{\lambda^{j_0}\chi^{j_1}}{j_0!}$$

and $c_{j_0|j_1}(W)$ is defined in Lemma 2.1.

Define the Λ-*bracket* $(a,b) \to [a_\Lambda b]$ of a local pair $\big(a(Z), b(Z)\big)$ by

$$[a_\Lambda b](W) := \mathcal{F}^{\Lambda}_{Z,W}[a(Z), b(W)].$$

Proposition 2.8 *The* Λ*-bracket satisfies the following properties for all pairwise local distributions* $(a(Z), b(Z), c(Z))$*:*

(1) (sesquilinearity)

$$[Sa_\Lambda b] = \chi[a_\Lambda b], \quad [a_\Lambda Sb] = -(-1)^{p(a)}(S+\chi)[a_\Lambda b];$$

(2) (skew symmetry)

$$[b_\Lambda a] = (-1)^{p(a)p(b)}[a_{-\Lambda-\nabla} b],$$

where

$$[a_{-\Lambda-\nabla} b] = \sum_{j_0\in\mathbb{Z}_{\geq 0},\, j_1=0,1} (-1)^{j_1}(-\Lambda-\nabla)^{(j_0|j_1)} a_{(j_0|j_1)} b$$

for $-\Lambda-\nabla = (-\lambda-T, -\chi-S)$ *with*

$$[\chi, S] = 2\lambda \quad \textit{and} \quad [\chi, T] = [\lambda, T] = [\lambda, S] = 0;$$

(3) (Jacobi identity)

$$[a_\Lambda[b_\Gamma c]] = -(-1)^{p(a)}[[a_\Lambda b]_{\Lambda+\Gamma} c] + (-1)^{(p(a)+1)(p(b)+1)}[b_\Gamma[a_\Lambda c]],$$

where

(i) $\Gamma = (\gamma, \eta)$ *with* $[\gamma, \eta] = [\gamma, \gamma] = 0$ *and* $[\eta, \eta] = -2\gamma$,
(ii) $\Lambda + \Gamma = (\lambda + \gamma, \zeta + \eta)$ *with* $[\lambda, \eta] = [\lambda, \gamma] = [\zeta, \gamma] = [\zeta, \eta] = 0$.

This motivates the following definition.

Definition 2.9 A *supersymmetric Lie conformal algebra (LCA)* $\mathcal{R}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathcal{K}$-module endowed with odd bilinear map $\mathcal{R} \otimes \mathcal{R} \to \mathcal{L} \otimes \mathcal{R}$, called Λ-bracket, given by a finite sum expansion

$$a \otimes b \mapsto [a_\Lambda b] = \sum_{j_0 \in \mathbb{Z}_{\geq 0},\ j_1 = 0,1} (-1)^{j_1} \Lambda^{(j_0|j_1)} a_{(j_0|j_1)} b$$

with $a_{(j_0|j_1)} b \in \mathcal{R}$, satisfying the following properties:

(1) (sesquilinearity) In $\mathcal{L} \otimes \mathcal{R}$ we have

$$[Sa_\Lambda b] = \chi [a_\Lambda b], \quad [a_\Lambda Sb] = -(-1)^{p(a)} (S + \chi)[a_\Lambda b]\,,$$

where S and χ obey the relation $[S, \chi] = 2\lambda$;

(2) (skew symmetry) In $\mathcal{L} \otimes \mathcal{R}$ we have

$$[b_\Lambda a] = (-1)^{p(a)p(b)} [a_{-\Lambda-\nabla} b],$$

where

$$[a_{-\Lambda-\nabla} b] = \sum_{j_0 \in \mathbb{Z}_{\geq 0},\ j_1 = 0,1} (-1)^{j_1} (-\Lambda - \nabla)^{(j_0|j_1)} a_{(j_0|j_1)} b$$

for $-\Lambda - \nabla = (-\lambda - T, -\chi - S)$ satisfying

$$[\chi, S] = 2\lambda \qquad \text{and} \qquad [\chi, T] = [\lambda, T] = [\lambda, S] = 0;$$

(3) (Jacobi identity) In $\mathcal{L} \otimes \mathcal{L}' \otimes \mathcal{R}$ we have

$$[a_\Lambda [b_\Gamma c]] = -(-1)^{p(a)} [[a_\Lambda b]_{\Lambda+\Gamma} c] + (-1)^{(p(a)+1)(p(b)+1)} [b_\Gamma [a_\Lambda c]],$$

where

(i) $\Gamma = (\gamma, \eta)$ such that $[\gamma, \eta] = [\gamma, \gamma] = 0$ and $[\eta, \eta] = -2\gamma$,
(ii) $\Lambda + \Gamma = (\lambda + \gamma, \zeta + \eta)$ such that $[\lambda, \eta] = [\lambda, \gamma] = [\zeta, \gamma] = [\zeta, \eta] = 0$.

Note that the tensor product sign is often omitted in the notation.

The next theorem provides an equivalent definition of supersymmetric vertex algebras in terms of Λ-brackets; cf. [20, Thm. 4.1].

Theorem 2.10 *A supersymmetric vertex algebra is a tuple* $(V, S, [\ _\Lambda\ ], |0\rangle, :\ \ :)$ *such that*

(i) $(V, S, [\ _\Lambda\ ])$ *is a supersymmetric Lie conformal algebra.*

(ii) $(V, S, |0\rangle, :\ :)$ *is a unital differential superalgebra, where* S *is an odd derivation of the product* $:\ :$*, and the following properties hold:*

$$
\begin{aligned}
&: ab : -(-1)^{p(a)p(b)} : ba := (-1)^{p(a)p(b)} \sum_{j \geq 1} \frac{(-T)^j}{j!} (b_{(-1+j|1)} a), \\
&:: ab : c : - : a : bc ::= \sum_{j \geq 0} a_{(-2-j|1)} (b_{(j|1)} c) \\
&\qquad + (-1)^{p(a)p(b)} \sum_{j \geq 0} b_{(-2-j|1)} (a_{(j|1)} c).
\end{aligned} \tag{2.2}
$$

(iii) The Λ*-bracket and the product* $:\ :$ *are related by the non-commutative Wick formula:*

$$
[a_\Lambda : bc :] = \sum_{k \geq 0} \frac{\lambda^k}{k!} [a_\Lambda b]_{(k-1|1)} c + (-1)^{(p(a)+1)p(b)} : b[a_\Lambda c] : . \tag{2.3}
$$

The properties (2.2) of the product $:\ :$ are referred to as the *quasi-commutativity* and *quasi-associativity*, respectively.

Definition 2.11 (1) A set $\mathcal{B} = \{a_i \mid i \in I\}$ of elements in a supersymmetric vertex algebra V *strongly generates* V if the set of monomials

$$
\{: a_{j_1} a_{j_2} \ldots a_{j_s} : \mid j_1, \ldots, j_s \in I,\ s \in \mathbb{Z}_{\geq 0}\}
$$

spans V. If $s = 0$, the monomial is understood as $|0\rangle$. For $s > 2$ the product in the monomial is applied consecutively from right to left.

(2) An ordered set $\mathcal{B} = \{a_i \mid i \in I\} \subset V$ *freely generates* a supersymmetric vertex algebra V if the set of monomials

$$
\{: a_{j_1} a_{j_2} \ldots a_{j_s} : \mid j_r \leq j_{r+1} \text{ and } j_r < j_{r+1} \text{ if } p(a_{j_r}) = \bar{1}\}
$$

forms a basis of V over $\mathbb{C}$.

Theorem 2.12 *Let* $\mathcal{R}$ *be a supersymmetric Lie conformal algebra with an ordered* $\mathbb{C}$*-basis* $\mathcal{B} = \{a_i \mid i \in I\}$*. Then, there exists a unique supersymmetric vertex algebra* $V(\mathcal{R})$ *such that*

(i) $V(\mathcal{R})$ *is freely generated by* $\mathcal{B}$*,*
(ii) the operator S *on* $V(\mathcal{R})$ *is defined by* $S(: ab :) =: (Sa)b : + (-1)^{p(a)} : a(Sb) :$*,*
(iii) the Λ*-bracket on* $\mathcal{R}$ *extends to the* Λ*-bracket on* $V(\mathcal{R})$ *via the Wick formula* (2.3).

Definition 2.13 For a given supersymmetric Lie conformal algebra $\mathcal{R}$, the supersymmetric vertex algebra $V(\mathcal{R})$ in Theorem 2.12 is called the *universal enveloping supersymmetric vertex algebra* associated to $\mathcal{R}$.

2.3 Supersymmetric nonlinear LCAs

In this section, we follow Section 3 of [9] to introduce *nonlinear* supersymmetric LCAs. We omit the arguments which are straightforward supersymmetric analogues of those in [9].

For a positive integer n, consider a $\mathcal{K}$-module $\mathcal{R} = \bigoplus_{\zeta\in\mathbb{N}/n}\mathcal{R}_\zeta$ with $(\mathbb{N}/n)$-grading so that $\mathrm{gr}(a) = \zeta$ for $a \in \mathcal{R}_\zeta$. The grading gr is naturally extended to the grading of the tensor algebra $\mathcal{T}(\mathcal{R})$ by

$$\mathrm{gr}(a\otimes b) = \mathrm{gr}(a) + \mathrm{gr}(b).$$

Set

$$\mathcal{T}(\mathcal{R})_{(\zeta)-} = \bigoplus_{\zeta'<\zeta}\mathcal{T}(\mathcal{R})_{\zeta'}.$$

Definition 2.14 Suppose that $\mathcal{R}$ is endowed with a *nonlinear* Λ*-bracket*

$$[\mathcal{R}_\zeta\,{}_\Lambda\,\mathcal{R}_{\zeta'}] \subset \mathcal{L}\otimes\mathcal{T}(\mathcal{R})_{(\zeta+\zeta')_-},$$

satisfying skew symmetry, sesquilinearity and Jacobi identity in Definition 2.9. Then, $\mathcal{R}$ is called *supersymmetric nonlinear Lie conformal algebra*.

Proposition 2.15 *Let $\mathcal{R}$ be a supersymmetric nonlinear LCA. Then, the normally ordered product and Λ-bracket admit unique extensions to the linear maps*

$$\mathcal{T}(\mathcal{R})\otimes\mathcal{T}(\mathcal{R})\to\mathcal{T}(\mathcal{R}),\quad A\otimes B\mapsto :AB:,$$
$$\mathcal{T}(\mathcal{R})\otimes\mathcal{T}(\mathcal{R})\to\mathcal{L}\otimes\mathcal{T}(\mathcal{R}),\quad A\otimes B\mapsto [A_\Lambda B],$$

in such a way that for any $a, b\in\mathcal{R}$ and $A, B, C\in\mathcal{T}(\mathcal{R})$ we have

(i) $[a_\Lambda b]$ *is defined by the* Λ*-bracket on* $\mathcal{R}$*,*
(ii) $:aB := a\otimes B$*,*
(iii) $:1A:=:A1:= A$*,*
(iv) $:(a\otimes B)C: - :a:BC::$ *is defined by the quasi-associativity,*
(v) $[A_\Lambda(b\otimes C)]$ *and* $[(a\otimes B)_\Lambda C]$ *are defined by the Wick formula.*

For a given supersymmetric nonlinear LCA $\mathcal{R}$, consider the two-sided ideal $\mathcal{J}(\mathcal{R})$ of $\mathcal{T}(\mathcal{R})$ generated by elements of the form

$$(:ab: - (-1)^{p(a)p(b)}:ba:) - (-1)^{p(a)p(b)}\sum_{j\geq 1}\frac{(-T)^j}{j!}b_{(-1+j|1)}a,$$

where

$$[b_\Lambda a] = \sum_{j_0\in\mathbb{Z}_{\geq 0},\, j_1=0,1}(-1)^{j_1}\Lambda^{(j_0|j_1)}b_{(j_0|j_1)}a.$$

Then, the Λ-bracket and the product $: \ :$ on $\mathcal{T}(\mathcal{R})$ induce a well-defined Λ-bracket and product on the quotient

$$V(\mathcal{R}) = \mathcal{T}(\mathcal{R})/\mathcal{J}(\mathcal{R}).$$

Since $V(\mathcal{R})$ satisfies quasi-commutativity, quasi-associativity and Wick formula, it is a supersymmetric vertex algebra which is called the *universal enveloping supersymmetric vertex algebra of* $\mathcal{R}$; cf. Definition 2.13.

Proposition 2.16 *For a given ordered basis $\mathcal{B}$ of $\mathcal{R}$, the supersymmetric vertex algebra $V(\mathcal{R})$ is freely generated by $\mathcal{B}$.*

3 Good filtered complexes of supersymmetric nonlinear LCAs

Here, we reproduce some useful facts about bigraded complexes. Proofs can be obtained by suitable supersymmetric versions of the arguments in [9, Sec. 4]. Introduce the notation

$$\Gamma = \frac{\mathbb{Z}}{2}, \quad \Gamma_+ = \frac{\mathbb{Z}_{\geq 0}}{2}, \quad \Gamma'_+ = \frac{\mathbb{Z}_{>0}}{2}.$$

Let $\mathfrak{g}$ be a graded vector superspace and $\mathcal{R} = \mathcal{K} \otimes \mathfrak{g}$ be a nonlinear Lie conformal algebra such that

$$\mathfrak{g} = \bigoplus_{\substack{p,q\in\Gamma,\ p+q=\mathbb{Z}_+, \\ \Delta\in\Gamma'_+}} \mathfrak{g}^{p,q}[\Delta], \qquad \mathcal{R} = \bigoplus_{\substack{p,q\in\Gamma,\ p+q=\mathbb{Z}_+, \\ \Delta\in\Gamma'_+}} \mathcal{R}^{p,q}[\Delta], \tag{3.1}$$

where

$$\mathcal{R}^{p,q}[\Delta] = \bigoplus_{n\geq 0} S^n \otimes \mathfrak{g}^{p,q}\big[\Delta - \frac{n}{2}\big].$$

The universal enveloping supersymmetric vertex algebra $V(\mathcal{R})$, which is strongly generated by a basis $\{a_i \mid i \in I\}$ of $\mathcal{R}$, has the Γ'_+-grading

$$V(\mathcal{R}) = \bigoplus_{\Delta\in\Gamma'_+} V(\mathcal{R})[\Delta]$$

where

$$V(\mathcal{R})[\Delta] = \mathrm{span}_{\mathbb{C}}\{: a_{i_1} a_{i_2} \dots a_{i_s} : \mid i_k \in I,\ a_{i_k} \in \mathcal{R}[\Delta_k],\ \textstyle\sum_{k=1}^{s} \Delta_k = \Delta\}.$$

We assume that

$$V(\mathcal{R})[\Delta_1]_{(n_0|n_1)} V(\mathcal{R})[\Delta_1] \subset V(\mathcal{R})\big[\Delta_1 + \Delta_2 - n_0 - \frac{n_1}{2} - \frac{1}{2}\big].$$

Consider a Γ-filtration and a $\mathbb{Z}$-grading of $\mathcal{R}$ induced from (3.1)

$$F^p\mathcal{R} = \bigoplus_{\substack{p'\geq p,\\ q,\Delta}} \mathcal{R}^{p',q}[\Delta], \qquad \mathcal{R}^n = \bigoplus_{p+q=n} \mathcal{R}^{p,q},$$

and the corresponding filtration and $\mathbb{Z}$-grading of $V(\mathcal{R})$ defined by

$$V(\mathcal{R})^n = \operatorname{span}_{\mathbb{C}}\{ : a_{i_1}a_{i_2}\dots a_{i_s} : \mid i_k \in I,\ a_{i_k} \in \mathcal{R}^{p_k,q_k},\ \textstyle\sum_{k=1}^s p_k + q_k = n\},$$
$$F^pV(\mathcal{R}) = \operatorname{span}_{\mathbb{C}}\{ : a_{i_1}a_{i_2}\dots a_{i_s} : \mid i_k \in I,\ a_{i_k} \in \mathcal{R}^{p_k,q_k},\ \textstyle\sum_{k=1}^s p_k \geq p\}.$$

Set

$$F^pV(\mathcal{R})^n = F^pV(\mathcal{R}) \cap V(\mathcal{R})^n, \quad F^pV(\mathcal{R})^n[\Delta] = F^pV(\mathcal{R})^n \cap V(\mathcal{R})[\Delta]$$

and consider the associated graded algebra

$$\operatorname{gr} V(\mathcal{R}) = \bigoplus_{p,q\in\Gamma} \operatorname{gr}^{p,q} V(\mathcal{R}),$$

where

$$\operatorname{gr}^{p,q} V(\mathcal{R})[\Delta] = F^pV(\mathcal{R})^{p+q}[\Delta]/F^{p+\frac{1}{2}}V(\mathcal{R})^{p+q}[\Delta],$$
$$\operatorname{gr}^{p,q} V(\mathcal{R}) = F^pV(\mathcal{R})^{p+q}/F^{p+\frac{1}{2}}V(\mathcal{R})^{p+q} = \bigoplus_{\Delta\in\Gamma'_+} \operatorname{gr}^{p,q} V(\mathcal{R})[\Delta].$$

Suppose a differential map $d : V(\mathcal{R}) \to V(\mathcal{R})$ satisfies

$$d(F^pV(\mathcal{R})^n) \subset F^pV(\mathcal{R})^{n+1}, \quad d(V(\mathcal{R}[\Delta]) \subset V(\mathcal{R})[\Delta]. \tag{3.2}$$

Then, we set for the cohomology spaces

$$F^pH^n(V(\mathcal{R}), d) = \operatorname{Ker}(d|_{F^pV(\mathcal{R})^n})/\operatorname{Im} d \cap F^pV(\mathcal{R})^n,$$
$$\operatorname{gr}^{p,q} H(V(\mathcal{R}), d) = F^pH^{p+q}(V(\mathcal{R}), d)/F^{p+\frac{1}{2}}H^{p+q}(V(\mathcal{R}), d).$$

In addition, for the graded differential map $d^{\operatorname{gr}} : \operatorname{gr} V(\mathcal{R}) \to \operatorname{gr} V(\mathcal{R})$ induced from d, we define cohomology spaces by

$$H^{p,q}(\operatorname{gr} V(\mathcal{R}), d^{\operatorname{gr}}) = \operatorname{Ker} d^{\operatorname{gr}}|_{\operatorname{gr}^{p,q} V(\mathcal{R})}/\operatorname{Im} d^{\operatorname{gr}} \cap \operatorname{gr}^{p,q} V(\mathcal{R}).$$

Definition 3.1 Let d be a differential on $V(\mathcal{R})$ satisfying (3.2).

(1) We say d is *almost linear differential of* $\mathcal{R}$ if

$$d^{\operatorname{gr}}(\mathfrak{g}^{p,q}[\Delta]) \subset \mathfrak{g}^{p,q+1}[\Delta];$$

or, equivalently, $d(\mathfrak{g}^{p,q}[\Delta]) \subset \mathfrak{g}^{p,q+1}[\Delta] \oplus F^{p+\frac{1}{2}}V(\mathcal{R})^{p+q+1}$.

(2) A differential d is called a *good* almost linear differential of $\mathcal{R}$ if

$$H^{p,q}(\mathfrak{g}, d^{\mathrm{gr}}) = 0 \quad \text{if} \quad p+q \neq 0.$$

In the rest of this section, we assume that $V(\mathcal{R})[\Delta]$ has finite dimension for any $\Delta \in \Gamma'_+$ and d is a good almost linear differential of $\mathcal{R}$. Take bases

$$\mathcal{B}^p_{\mathfrak{g}}[\Delta] = \{\, e_i \mid i \in \mathcal{I}^p_{\mathfrak{g}}[\Delta]\,\} \quad \text{for some index sets } \mathcal{I}^p_{\mathfrak{g}}[\Delta],$$
$$\mathcal{B}^p_{\mathcal{R}}[\Delta] = \{e_{(i,n)} \mid e_{(i,n)} = S^n e_i,\ e_i \in \mathcal{B}^p_{\mathfrak{g}}[\Delta'],\ \Delta' + \frac{n}{2} = \Delta\},$$

of $\mathfrak{g}^{p,-p}[\Delta] \cap \operatorname{Ker} d^{\mathrm{gr}}$ and $\mathcal{R}^{p,-p}[\Delta] \cap \operatorname{Ker} d^{\mathrm{gr}} = H^{p,-p}(\mathrm{gr}\,\mathcal{R}, d^{\mathrm{gr}})[\Delta]$, respectively. Then,

$$\mathcal{B}_{\mathcal{R}} := \bigsqcup_{\Delta \in \Gamma'_+,\ p \in \Gamma} \mathcal{B}^p_{\mathcal{R}}[\Delta] = \{\, e_{(i,n)} \mid e_{(i,n)} = S^n e_i,\ i \in \mathcal{I}_{\mathfrak{g}}\,\}$$

is a basis of $H(\mathrm{gr}\,\mathcal{R}, d^{\mathrm{gr}})$, where

$$\mathcal{I}_{\mathfrak{g}} := \bigsqcup_{\Delta \in \Gamma'_+,\ p \in \Gamma} \mathcal{I}^p_{\mathfrak{g}}[\Delta].$$

Proposition 3.2 *(1) $H(gr\, V(\mathcal{R}), d^{gr})$ is freely generated by $\mathcal{B}_{\mathcal{R}}$.*
(2) $H^{p,-p}(gr\, V(\mathcal{R}), d^{gr})[\Delta]$ has the basis

$$\mathcal{B}^p_{V(\mathcal{R})}[\Delta] = \Big\{ :e_{(i_1,n_1)} e_{(i_2,n_2)} \dots e_{(i_k,n_k)}: \Big\},$$

where the sets of indices $(i_t, n_t) \in \mathcal{I}^{p_t}_{\mathfrak{g}}[\Delta_t] \times \mathbb{Z}_{\geq 0}$ satisfy the conditions:

(i) $(i_t, n_t) \leq (i_{t+1}, n_{t+1})$,
(ii) if $e_{(i_t,n_t)}$ and $e_{(i_{t+1},n_{t+1})}$ are odd, then $(i_t, n_t) < (i_{t+1}, n_{t+1})$,
(iii) $\sum_{t=1}^k i_t = p$,
(iv) $\sum_{t=1}^k \left(\Delta_t + \frac{n_t}{2}\right) = \Delta$.

For $e_i \in \mathfrak{g}^{p,-p}[\Delta] \cap \operatorname{Ker} d^{\mathrm{gr}}$, there exists an element $f_i \in F^{p+\frac{1}{2}}V(\mathcal{R})^0[\Delta]$ such that $E_i = e_i + f_i \in F^p V(\mathcal{R})^0[\Delta] \cap \operatorname{Ker} d$. Set

$$H^{p,-p}(\mathfrak{g}, d)[\Delta] = \operatorname{span}\{\, E_i \mid i \in \mathcal{I}^p_{\mathfrak{g}}[\Delta]\,\}, \quad H(\mathfrak{g}, d)[\Delta] = \bigoplus_{p \in \Gamma} H^{p,-p}(\mathfrak{g}, d)[\Delta].$$

Theorem 3.3 *(1) $H(V(\mathcal{R}), d) = H^0(V(\mathcal{R}), d)$.*
(2) If the $\mathcal{K}$-module $H(\mathcal{R}, d) = \mathcal{K} \otimes H(\mathfrak{g}, d)$ admits a nonlinear supersymmetric LCA structure, then

$$H(V(\mathcal{R}), d) \simeq V(H(\mathcal{R}, d)).$$

4 BRST cohomology

We are now in a position to define supersymmetric W-algebras via BRST cohomology following [23]. We will rely on the supersymmetric vertex algebra theory developed by Heluani and Kac [16,19] to describe the structure of the W-algebras associated with odd nilpotent elements of Lie superalgebras.

4.1 BRST complex

Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra with a $(\frac{1}{2}\mathbb{Z})$-grading $\mathfrak{g} = \bigoplus_{i\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}(i)$ satisfying the following conditions:

(i) There exists $h \in \mathfrak{g}_{\bar{0}}$ such that $\mathfrak{g}(i) = \{a \in \mathfrak{g} \,|\, \frac{1}{2}[h, a] = ia\}$.
(ii) There are odd elements $f_{\mathrm{odd}} \in \mathfrak{g}(-\frac{1}{2})$ and $e_{\mathrm{odd}} \in \mathfrak{g}(\frac{1}{2})$ such that

$$\mathrm{span}\{e, e_{\mathrm{odd}}, h, f_{\mathrm{odd}}, f\} \simeq \mathfrak{osp}(1|2),$$

where (e, h, f) is an $\mathfrak{sl}_2$-triple.

We will suppose that $\mathfrak{g}$ is equipped with a non-degenerate invariant bilinear form $(\,|\,)$ normalized by the conditions $(e|f) = \frac{1}{2}(h|h) = 1$.

Introduce two supersymmetric vertex algebras.

(1) Let $\bar{\mathfrak{g}} = \{\bar{a} \,|\, a \in \mathfrak{g}\}$ be the vector superspace defined by $\bar{\mathfrak{g}}_{\bar{1}} = \mathfrak{g}_{\bar{0}}$ and $\bar{\mathfrak{g}}_{\bar{0}} = \mathfrak{g}_{\bar{1}}$. The *supersymmetric current nonlinear LCA* is

$$\mathcal{R}_{\mathrm{cur}} := \mathcal{K} \otimes \bar{\mathfrak{g}}$$

endowed with the Λ-bracket

$$[\bar{a}_\Lambda \bar{b}] = (-1)^{p(a)p(\bar{b})}\overline{[a, b]} + k\,\chi(a|b).$$

(2) Set $\mathfrak{n} = \bigoplus_{i>0} \mathfrak{g}(i)$ and $\mathfrak{n}_- = \bigoplus_{i<0} \mathfrak{g}(i)$. Then, there are bases

$$\{u_\alpha \,|\, \alpha \in I_+\} \quad \text{and} \quad \{u^\alpha \,|\, \alpha \in I_+\}$$

of $\mathfrak{n}$ and $\mathfrak{n}_-$, respectively, parameterized by a certain index set I_+, such that $(u^\alpha|u_\beta) = \delta_{\alpha,\beta}$. Introduce two vector superspaces

$$\phi_{\mathfrak{n}} \simeq \mathfrak{n} \subset \mathfrak{g}, \qquad \phi^{\bar{\mathfrak{n}}_-} \simeq \bar{\mathfrak{n}}_- \subset \bar{\mathfrak{g}},$$

spanned by the respective families of elements ϕ_b and $\phi^{\bar{a}}$ with $b \in \mathfrak{n}$ and $\bar{a} \in \bar{\mathfrak{n}}_-$. Consider the supersymmetric nonlinear LCA $\mathcal{R}_{\mathrm{ch}} = \mathcal{K} \otimes (\phi_{\mathfrak{n}} \oplus \phi^{\bar{\mathfrak{n}}_-})$ endowed with the Λ-bracket

$$[\phi^{\bar{a}}\,{}_\Lambda \phi_b] = [\phi_b\,{}_\Lambda \phi^{\bar{a}}] = (a|b).$$

Due to the results of Sect. 2.3, the two above supersymmetric nonlinear LCAs give rise to respective universal enveloping supersymmetric vertex algebras $V(\mathcal{R}_{\text{cur}})$ and $V(\mathcal{R}_{\text{ch}})$. Their tensor product

$$C(\overline{\mathfrak{g}}, f_{\text{odd}}, k) = V(\mathcal{R}_{\text{cur}}) \otimes V(\mathcal{R}_{\text{ch}})$$

also carries a supersymmetric vertex algebra structure. Introduce the element d by

$$d = \sum_{\alpha \in I_+} :(\overline{u}_\alpha - (f_{\text{odd}}|u_\alpha))\phi^\alpha: + \frac{1}{2} \sum_{\alpha,\beta \in I_+} (-1)^{p(\alpha)p(\overline{\beta})} :\phi_{[u_\alpha, u_\beta]}\phi^\beta \phi^\alpha:, \quad (4.1)$$

where $\phi^\alpha = \phi^{\overline{u}^\alpha}$, $\phi_\alpha = \phi_{u_\alpha}$, $p(\alpha) = p(u_\alpha)$ and $p(\overline{\alpha}) = p(\overline{u}_\alpha)$.

Proposition 4.1 *The Λ-brackets between d and elements in $C(\overline{\mathfrak{g}}, f_{odd}, k)$ have the form:*

$$[d_\Lambda \overline{a}] = \sum_{\alpha \in I_+} (-1)^{p(\overline{\alpha})p(\alpha)} :\phi^\alpha \overline{[u_\alpha, a]}: + \sum_{\alpha \in I_+} (-1)^{p(\overline{\alpha})} k(\chi + S)\phi^\alpha (u_\alpha|a),$$

$$[d_\Lambda \phi^\alpha] = \frac{1}{2} \sum_{\alpha,\beta \in I_+} (-1)^{p(\overline{\alpha})p(\beta)} :\phi^\beta \phi^{\overline{[u_\beta, u^\alpha]}}:,$$

$$[d_\Lambda \phi_\alpha] = (-1)^{p(\overline{\alpha})} u_\alpha - (f_{odd}|u_\alpha) + \sum_{\beta \in I_+} (-1)^{p(\overline{\alpha})p(\beta)} :\phi^\beta \phi_{[u_\beta, u_\alpha]}:.$$

Proof The formulas are verified by a direct calculation in the same way as for the supersymmetric classical W-algebras; see [25]. □

Set $Q := d_{(0|0)}$. Then, by the Wick formula (2.3), we have

$$Q(:A\,B:) =: Q(A)\,B: + (-1)^{p(A)} :A\,Q(B):. \quad (4.2)$$

Proposition 4.2 *The linear map Q on $C(\overline{\mathfrak{g}}, f_{odd}, k)$ satisfies $Q^2 = 0$.*

Proof This follows by a direct computation with the use of Proposition 4.1 and property (4.2). □

By taking the cohomology of the BRST complex $C(\overline{\mathfrak{g}}, f_{\text{odd}}, k)$ with the differential Q, we can now define the corresponding supersymmetric W-algebra as in [23]; cf. [2] and [15, Ch. 15].

Definition 4.3 The *supersymmetric W-algebra associated to $\overline{\mathfrak{g}}$, f_{odd} and $k \in \mathbb{C}$* is

$$W(\overline{\mathfrak{g}}, f_{\text{odd}}, k) = H(C(\overline{\mathfrak{g}}, f_{\text{odd}}, k), Q).$$

Proposition 4.4 *Let $A, B \in C(\overline{\mathfrak{g}}, f_{odd}, k)$ satisfy $Q(A) = Q(B) = 0$ and C be any element in $C(\overline{\mathfrak{g}}, f_{odd}, k)$. Then, the following holds:*

(1) $Q(SA) = Q(:AB:) = 0$ *and* $Q([A_\Lambda B]) = 0$*;*
(2) $S(QC)$, $:Q(C)\,B:$ *and* $[Q(C)_\Lambda B]$ *belong to the image of* Q.

Proof By sesquilinearity of supersymmetric LCAs, for any $X \in C(\bar{\mathfrak{g}}, f_{\mathrm{odd}}, k)$ we have $S(QX) = -Q(SX)$. Hence, the first properties in (1) and (2) hold. The second properties follow from (4.2). By the Jacobi identity of supersymmetric LCAs, for $X, Y \in C(\bar{\mathfrak{g}}, f_{\mathrm{odd}}, k)$ we have

$$Q([X_\Lambda Y]) = -[Q(X)_\Lambda Y] + (-1)^{p(X)+1}[X_\Lambda Q(Y)]$$

which gives the third properties in (1) and (2). □

Corollary 4.5 *The supersymmetric W-algebra* $W(\bar{\mathfrak{g}}, f_{odd}, k)$ *is a supersymmetric vertex algebra.*

4.2 Building blocks of supersymmetric *W*-algebras

For any $\bar{a} \in \bar{\mathfrak{g}}$ set

$$J_{\bar{a}} = \bar{a} + \sum_{\beta\in I_+}(-1)^{p(\bar{a})p(\bar{\beta})} :\phi^\beta \phi_{[u_\beta,a]}: \in C(\bar{\mathfrak{g}}, f_{\mathrm{odd}}, k).$$

Proposition 4.6 *For the element d defined in* (4.1)*, we have*

$$\begin{aligned}[d_\Lambda J_{\bar{a}}] = &\sum_{\beta\in I_+}(-1)^{p(\bar{a})p(\beta)} :\phi^\beta(J_{\overline{\pi_{\le 0}[u_\beta,a]}} + (f_{odd}|[u_\beta,a])): \\ &+ \sum_{\beta\in I_+}(-1)^{\overline{\beta}} k(S+\chi)\phi^\beta(u_\beta|a),\end{aligned}$$

where $\pi_{\le 0} : \mathfrak{g} \to \oplus_{i\le 0}\mathfrak{g}(i)$ *is the projection map with the kernel* $\oplus_{i>0}\mathfrak{g}(i)$.

Proof By the Wick formula,

$$\begin{aligned}[d_\Lambda J_{\bar{a}}] &= [d_\Lambda \bar{a}] + \sum_{\beta\in I_+}(-1)^{p(\bar{a})p(\bar{\beta})}[d\,{}_\Lambda :\phi^\beta\phi_{[u_\beta,a]}:] \\ &= [d_\Lambda \bar{a}] + \sum_{\beta\in I_+}(-1)^{p(\bar{a})p(\bar{\beta})} :[d\,{}_\Lambda \phi^\beta]\phi_{[u_\beta,a]}: \qquad (4.3) \\ &\quad + \sum_{\beta,\gamma\in I_+,\,k\ge 1}\frac{\lambda^k}{2k!}(-1)^{p(\bar{\beta})(p(\gamma)+p(a)+1)}\left(:\phi^\gamma\phi^{\overline{[u_\gamma,u^\beta]}}:\right)_{(k-1|1)}\phi_{[u_\beta,a]} \\ &\quad + \sum_{\beta\in I_+}(-1)^{p(\bar{a})p(\bar{\beta})} :\phi^\beta[d_\Lambda\phi_{[u_\beta,a]}]:. \qquad (4.4)\end{aligned}$$

Since the coefficients of $\Lambda^{j_0}\chi$ in $[\phi_{[u_\beta,a]}\,{}_\Lambda\, :\phi^\gamma\phi^{\overline{[u_\gamma,u^\beta]}}:]$ are all zero, the coefficients of $\Lambda^{j_0}\chi$ in

$$[:\phi^\gamma\phi^{\overline{[u_\gamma,u^\beta]}}:\ {}_\Lambda\phi_{[u_\beta,a]}] = (-1)^{p(\beta)p(\bar{a})}[\phi_{[u_\beta,a]}\ {}_{-\Lambda-\nabla}\ :\phi^\gamma\phi^{\overline{[u_\gamma,u^\beta]}}:]$$

are also 0 so that the expression in (4.4) vanishes. The second term in (4.3) equals

$$\sum_{\beta,\gamma\in I_+}\frac{1}{2}(-1)^{p(\bar{\beta})(p(\gamma)+p(a)+1)}::\phi^\gamma\phi^{\overline{[u_\gamma,u^\beta]}}:\ \phi_{[u_\beta,a]}:.$$

By the quasi-associativity in (2.2) and the fact that $\phi^{\bar{n}}\,{}_{(j|1)}\phi_m = 0$ for any $n\in\mathfrak{n}$ and $m\in\mathfrak{n}_-$ with $j\geq 0$, we have

$$::\phi^\gamma\phi^{\overline{[u_\gamma,u^\beta]}}:\phi_{[u_\beta,a]}:=:\phi^\gamma\ :\phi^{\overline{[u_\gamma,u^\beta]}}\phi_{[u_\beta,a]}::\ .$$

The remaining computations are straightforward, and they are analogous to the classical case in [25]. □

Proposition 4.7 *If $a,b\in\bigoplus_{i\leq 0}\mathfrak{g}(i)$ or $a,b\in\bigoplus_{i>0}\mathfrak{g}$, then*

$$[J_{\bar{a}}\,{}_\Lambda\, J_{\bar{b}}] = (-1)^{p(a)p(\bar{b})}J_{\overline{[a,b]}} + k\,(S+\chi)\,(a|b).$$

Proof This is verified by a direct computation. □

Introduce the vector superspaces

$$r_+ = \phi_{\mathfrak{n}}\oplus J_{\bar{\mathfrak{n}}} \qquad \text{and} \qquad r_- = J_{\bar{\mathfrak{g}}_{\leq 0}}\oplus\phi^{\bar{\mathfrak{n}}_-},$$

where

$$J_{\bar{\mathfrak{n}}} = \text{span}\,\{J_b\,|\,b\in\bar{\mathfrak{n}}\} \qquad \text{and} \qquad J_{\bar{\mathfrak{g}}_{\leq 0}} = \text{span}\,\{J_{\bar{a}}\,|\,a\in\bigoplus_{i\in\mathbb{Z}_{\leq 0}}\mathfrak{g}(i)\}.$$

It is not difficult to see that both $\mathcal{R}_+ = \mathcal{K}\otimes r_+$ and $\mathcal{R}_- = \mathcal{K}\otimes r_-$ are supersymmetric nonlinear LCAs and that $C(\bar{\mathfrak{g}}, f_{\text{odd}}, k)$ decomposes into the tensor product of supersymmetric vertex subalgebras:

$$C(\bar{\mathfrak{g}}, f_{\text{odd}}, k) = V(\mathcal{R}_+)\otimes V(\mathcal{R}_-).$$

Lemma 4.8 *(Künneth lemma) Let V_1 and V_2 be vector superspaces and $d_i : V_i\to V_i$, $i=1,2$, be differentials. If $d: V_1\otimes V_2\to V_1\otimes V_2$ is defined by*

$$d(a\otimes b) = d_1(a)\otimes b + (-1)^{p(a)}a\otimes d_2(b),$$

then

$$H(V, d) \simeq H(V_1, d_1) \otimes H(V_2, d_2).$$

Proposition 4.9 *The differential Q has the properties*

$$Q(V(\mathcal{R}_+)) \subset V(\mathcal{R}_+) \quad \textit{and} \quad Q(V(\mathcal{R}_-)) \subset V(\mathcal{R}_-), \tag{4.5}$$

so that

$$W(\bar{\mathfrak{g}}, f_{odd}, k) = H(V(\mathcal{R}_+), Q) \otimes H(V(\mathcal{R}_-), Q). \tag{4.6}$$

Proof The inclusions (4.5) follow from Propositions 4.1 and 4.6. The decomposition (4.6) is then implied by the Künneth lemma. □

4.3 Generators of supersymmetric *W*-algebras

We now aim to describe the cohomologies $H(V(\mathcal{R}_+), Q)$ and $H(V(\mathcal{R}_-), Q)$.

Proposition 4.10 *We have $H(V(\mathcal{R}_+), Q) = \mathbb{C}$ so that $W(\bar{\mathfrak{g}}, f_{odd}, k) = H(V(\mathcal{R}_-), Q)$.*

Proof Set $K_{\bar{n}} = (-1)^{p(\bar{n})} J_{\bar{n}} - (f_{\text{odd}}|n)$ for $n \in \mathfrak{n}$ and introduce the superspace

$$r'_+ = \phi_{\mathfrak{n}} \oplus K_{\bar{\mathfrak{n}}}, \qquad K_{\bar{\mathfrak{n}}} = \text{span}\,\{K_{\bar{n}} \,|\, \bar{n} \in \bar{\mathfrak{n}}\}.$$

Then, $\mathcal{R}_+ = \mathcal{K} \otimes r'_+$. Define the conformal weight Δ and the bigrading on r'_+ by

$$\Delta(\phi_n) = \Delta(K_{\bar{n}}) = j_n, \quad \text{gr}(\phi_n) = (j_n - 1, -j_n), \quad \text{gr}(K_{\bar{n}}) = (j_n - 1, -j_n + 1),$$

assuming that $n \in \mathfrak{g}(j_n)$. The graded differential Q^{gr} associated with Q is good almost linear (see Sect. 3) and

$$H({r'}_+, Q^{\text{gr}}) = 0.$$

By Theorem 3.3, we have $H(V(\mathcal{R}_+), Q) = \mathbb{C}$. □

To describe $H(V(\mathcal{R}_-), Q)$, recall that

$$\begin{aligned} Q(J_{\bar{a}}) = \sum_{\beta \in I_+} (-1)^{p(\bar{a})p(\beta)} :\phi^\beta (J_{\overline{\pi_{\leq 0}[u_\beta, a]}} + (f_{\text{odd}}|[u_\beta, a])): \\ + \sum_{\beta \in I_+} (-1)^{p(\bar{\beta})} k S\phi^\beta (u_\beta|a) \end{aligned} \tag{4.7}$$

and

$$Q(\phi^{\bar{m}}) = \frac{1}{2} \sum_{\beta \in I_+} (-1)^{p(\bar{m})p(\beta)} :\phi^\beta \phi^{\overline{[u_\beta, m]}}: . \tag{4.8}$$

Consider the conformal weight Δ and the bigrading on r_- satisfying

$$\Delta(J_{\bar{a}}) = \frac{1}{2} - j_a, \quad \Delta(\phi^{\bar{m}}) = -j_m,$$
$$\mathrm{gr}(J_{\bar{a}}) = (j_a, -j_a), \quad \mathrm{gr}(\phi^{\bar{m}}) = \big(j_m + \frac{1}{2}, -j_m + \frac{1}{2}\big),$$

where $a \in \mathfrak{g}(j_a)$ and $m \in \mathfrak{g}(j_m)$ for $j_a \leq 0$ and $j_m < 0$. Note that

$$\Delta(\phi^\beta) = j_\beta, \quad \mathrm{gr}(\phi^\beta) = \big(-j_\beta + \frac{1}{2}, j_\beta + \frac{1}{2}\big),$$

where $u^\beta \in \mathfrak{g}(-j_\beta)$. Since $\Delta(S) = \frac{1}{2}$ and $\mathrm{gr}(S) = (0, 0)$, every term in (4.7) has conformal weight $\frac{1}{2} - j_a$ and every term in (4.8) has conformal weight $-j_m$. The bigradings of terms in (4.7) are given by

$$\begin{aligned}
&\mathrm{gr}(:\phi^\beta J_{\overline{\pi_{\leq 0}[u_\beta, a]}}:) = \big(j_a + \frac{1}{2}, -j_a + \frac{1}{2}\big),\\
&\mathrm{gr}(\phi^\beta(f_{\mathrm{odd}}|[u_\beta, a])) = (j_a, -j_a + 1),\\
&\mathrm{gr}(S\phi^\beta(u_\beta|a)) = \big(j_a + \frac{1}{2}, -j_a + \frac{1}{2}\big).
\end{aligned} \tag{4.9}$$

The bigradings of terms in (4.8) are

$$\mathrm{gr}(\phi^{\bar{m}}) = \big(j_m + \frac{1}{2}, -j_m + \frac{1}{2}\big), \qquad \mathrm{gr}(:\phi^\beta \phi^{\overline{[u_\beta, m]}}:) = (j_m + 1, -j_m + 1). \tag{4.10}$$

Theorem 4.11 *Let* Ker (ad f_{odd}) $= \{ u_\alpha \,|\, \alpha \in \mathcal{J} \}$ *with an index set* $\mathcal{J}$. *Then,*

(1) $W(\bar{\mathfrak{g}}, f_{odd}, k)$ is freely generated by $|\mathcal{J}|$ elements as a differential algebra,
(2) there exists a free generating set of the form

$$\{ u_\alpha + A_\alpha \,|\, \alpha \in \mathcal{J} \},$$

where $A_\alpha \in F^{j_\alpha + \frac{1}{2}} V(\mathcal{R}_-)^0[\frac{1}{2} - j_\alpha]$ *for* $u_\alpha \in \mathfrak{g}(j_\alpha)$.

Proof Since we know that $W(\bar{\mathfrak{g}}, f_{\mathrm{odd}}, k) = H(V(\mathcal{R}_-), Q)$, it is enough to show (1) and (2) for $H(V(\mathcal{R}_-), Q)$. The conformal weight and bigrading on r_- induce those on $V(\mathcal{R}_-)$. With respect to the conformal weight and bigrading, Q induces the graded differential Q^{gr}. The bigradings listed in (4.9) and (4.10) show that

$$Q^{\mathrm{gr}}(J_{\bar{a}}) = \sum_{\beta \in I_+} (-1)^{p(\bar{a})p(\beta)} \phi^\beta (f_{\mathrm{odd}}|[u_\beta, a]), \quad Q^{\mathrm{gr}}(\phi^{\bar{m}}) = 0.$$

Note that $V(\mathcal{R}_-)^0 \cap r_- = J_{\mathfrak{g}_{\leq 0}}$ and $V(\mathcal{R}_-)^1 \cap r_- = \phi^{\bar{\mathfrak{n}}_-}$. Since $Q^{\mathrm{gr}}(r_-) = \phi^{\bar{\mathfrak{n}}_-}$, we have $H^{p,q}(r_-, Q^{\mathrm{gr}}) = 0$ when $p+q \neq 0$ and so Q is a good almost linear differential map. Furthermore, $\mathrm{Ker}(Q^{\mathrm{gr}}|_{r_-}) = \{J_a | a \in \mathrm{Ker}(\mathrm{ad}\, f_{\mathrm{odd}})\} \oplus \phi^{\bar{\mathfrak{n}}_-}$; hence,

$$H(r_-, Q^{\mathrm{gr}}) = \{J_a \mid a \in \mathrm{Ker}(\mathrm{ad}\, f_{\mathrm{odd}})\}.$$

Thus, using Theorem 3.3, we arrive at (1) and (2). □

5 Generators of $W(\bar{\mathfrak{g}}, f_{\mathrm{prin}}, k)$ for $\mathfrak{g} = \mathfrak{gl}(n+1|n)$

Consider the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(n+1|n)$ with the basis $\{E_{i,j} | i, j = 1, \ldots, 2n+1\}$ and the $\mathbb{Z}/2\mathbb{Z}$-grading defined by $p(E_{i,j}) = i + j \mod 2$ with the commutation relations

$$[E_{i,j}, E_{i',j'}] = \delta_{j,i'} E_{i,j'} - (-1)^{(i+j)(i'+j')}\, \delta_{i,j'}\, E_{i',j}.$$

Take the odd principal nilpotent element in the form

$$f_{\mathrm{prin}} = \sum_{p=1}^{2n} E_{p+1,p}.$$

By Proposition 4.6, for $C(\bar{\mathfrak{g}}, f_{\mathrm{prin}}, k)$ and any $m \geq l$, we have

$$\begin{aligned} Q(J_{m,l}) = {} & (-1)^m k S \phi^{l,m} + \sum_{j=l+1}^{m} (-1)^{l+j+1} : \phi^{l,j} J_{m,j} : \\ & + \sum_{i=l}^{m-1} (-1)^{(i+m)(m+l+1)} : \phi^{i,m} J_{i,l} : + (-1)^l \phi^{l,m+1} + (-1)^m \phi^{l-1,m}, \end{aligned}$$

where we set $\phi^{j,i} = (-1)^{i+1} \phi^{\overline{E_{ij}}}$ for $i > j$ and $J_{i,j} = J_{\overline{E_{i,j}}}$ for $i \geq j$.

We will be working with operators on $C(\bar{\mathfrak{g}}, f_{\mathrm{prin}}, k)$ of the form $\sum_{t=0}^{N} A_t S^t$ with $A_t \in C(\bar{\mathfrak{g}}, f_{\mathrm{prin}}, k)$, which act on an arbitrary element $X \in C(\bar{\mathfrak{g}}, f_{\mathrm{prin}}, k)$ by the rule

$$\sum_{t=0}^{N} A_t S^t(X) = \sum_{t=0}^{N} : A_t(S^t(X)) : .$$

In particular, for the operator $A_{i,j} = \delta_{ij} k S + (-1)^{i+1} J_{i,j}$ on $C(\bar{\mathfrak{g}}, f_{\mathrm{prin}}, k)$ we have

$$A_{i,j}(X) = \delta_{ij} k S(X) + (-1)^{i+1} : J_{i,j} X : .$$

Consider the $(2n+1)\times(2n+1)$ matrix

$$\mathcal{A} := \begin{bmatrix} A_{1,1} & -1 & 0 & \cdots & \cdots & 0 \\ A_{2,1} & A_{2,2} & -1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{2n,1} & A_{2n,2} & A_{2n,3} & \cdots & A_{2n,2n} & -1 \\ A_{2n+1,1} & A_{2n+1,2} & A_{2n+1,3} & \cdots & A_{2n+1,2n} & A_{2n+1,2n+1} \end{bmatrix}$$

whose entries are operators on $C(\bar{\mathfrak{g}}, f_{\text{prin}}, k)$. Then, the *column* (or *row*) *determinant* of $\mathcal{A}$ is given by the formula

$$\operatorname{cdet}\mathcal{A} = \sum_{N=0}^{2n} \sum_{0=i_0<i_1<\cdots<i_{N+1}=2n+1} A_{i_1,i_0+1}A_{i_2,i_1+1}\dots A_{i_{N+1},i_N+1}. \tag{5.1}$$

Write

$$\operatorname{cdet}\mathcal{A} = W_0 + W_1 S + \cdots + W_{2n+1}S^{2n+1}$$

for certain elements $W_p \in C(\bar{\mathfrak{g}}, f_{\text{prin}}, k)$. Clearly, $W_{2n+1} = k^{2n+1}$.

Theorem 5.1 *All elements $W_1, \dots, W_{2n}$ belong to the W-algebra $W(\bar{\mathfrak{g}}, f_{prin}, k)$.*

Proof One readily verifies that

$$Q\sum_{p=0}^{2n+1} W_p S^p = \sum_{p=0}^{2n+1} Q(W_p)S^p - W_p S^p Q$$

so that $QA_{m,l} = (-1)^{m+l+1}A_{m,l}Q + (-1)^{m+1}Q(J_{m,l})$. Therefore,

$$\begin{aligned} &Q\, A_{i_1,i_0+1}\dots A_{i_{p+1},i_p+1}\dots A_{i_{N+1},i_N+1} \\ &\quad= \sum_{p=0}^{N}(-1)^{i_p}\big(A_{i_1,i_0+1}\dots\big((-1)^{i_{p+1}+1}Q(J_{i_{p+1},i_p+1})\big)\dots A_{i_{N+1},i_N+1}\big) \\ &\qquad\quad -A_{i_1,i_0+1}\dots A_{i_{p+1},i_p+1}\dots A_{i_{N+1},i_N+1}Q. \end{aligned}$$

Hence, the property $W_p \in W(\mathfrak{g}, f_{\text{prin}}, k)$ will follow if we show that $\sum_{N=0}^{2n} B_N = 0$, where we set

$$B_N = \sum_{p=0}^{N}(-1)^{i_p}\big(A_{i_1,i_0+1}\dots\big((-1)^{i_{p+1}+1}Q(J_{i_{p+1},i_p+1})\big)\dots A_{i_{N+1}i_N+1}\big).$$

Using the relations

$$J_{i,j} = (-1)^{i+1}(A_{i,j} - \delta_{i,j}kS) \quad\text{and}\quad :\phi^{j,i}J_{i',j'} := (-1)^{(i+j+1)(i'+j'+1)} : J_{i',j'}\phi^{j,i}:$$

we find that

$$
\begin{aligned}
&(-1)^{i_{p+1}+1} Q(J_{i_{p+1},i_p+1}) \\
&\qquad = -kS(\phi^{i_p+1,i_{p+1}}) + \sum_{j=i_p+2}^{i_{p+1}} (-1)^{i_p+j} \phi^{i_p+1,j} \big(A_{i_{p+1},j} - \delta_{i_{p+1},j} kS\big) \\
&\quad + \sum_{i=i_p+1}^{i_{p+1}-1} (-1)^{i_p+i} \big(A_{i,i_p+1} - \delta_{i,i_p+1} kS\big) \phi^{i,i_{p+1}} + (-1)^{i_p+i_{p+1}} \phi^{i_p+1,i_{p+1}+1} - \phi^{i_p,i_{p+1}}
\end{aligned}
$$

and

$$
-kS(\phi^{i_p+1,i_{p+1}}) + (-1)^{i_p+i_{p+1}+1} \phi^{i_p+1,i_{p+1}} S + S\phi^{i_p+1,i_{p+1}} = 0.
$$

Therefore,

$$
\begin{aligned}
(-1)^{i_{p+1}+1} Q(J_{i_{p+1},i_p+1}) = & \sum_{j=i_p+2}^{i_{p+1}} (-1)^{i_p+j} \phi^{i_p+1,j} A_{i_{p+1},j} \\
& + \sum_{i=i_p+1}^{i_{p+1}-1} (-1)^{i_p+i} A_{i,i_p+1} \phi^{i,i_{p+1}} + (-1)^{i_p+i_{p+1}} \phi^{i_p+1,i_{p+1}+1} - \phi^{i_p,i_{p+1}}
\end{aligned}
$$

so that B_N can be expressed as

$$
\begin{aligned}
&\sum_{p=0}^{N} A_{i_1,i_0+1} \dots A_{i_p,i_{p-1}+1} \Bigg[\Bigg(\sum_{j=i_p+2}^{i_{p+1}} (-1)^j \phi^{i_p+1,j} A_{i_{p+1},j} + (-1)^{i_{p+1}} \phi^{i_p+1,i_{p+1}+1} \Bigg) \\
&+ \Bigg(\sum_{i=i_p+1}^{i_{p+1}-1} (-1)^i A_{i,i_p+1} \phi^{i,i_{p+1}} - (-1)^{i_p} \phi^{i_p,i_{p+1}} \Bigg) \Bigg] A_{i_{p+2},i_{p+1}+1} \dots A_{i_{N+1},i_N+1}.
\end{aligned}
$$

By the quasi-associativity property, we have

$$
\begin{aligned}
(\phi^{i_p+1,j} A_{i_{p+1},j})(A_{i_{p+2},i_{p+1}+1} \dots A_{i_{N+1},i_N+1}) &= \phi^{i_p+1,j} (A_{i_{p+1},j} (A_{i_{p+2},i_{p+1}+1} \dots A_{i_{N+1},i_N+1})), \\
(A_{i,i_p+1} \phi^{i,i_{p+1}})(A_{i_{p+2},i_{p+1}+1} \dots A_{i_{N+1},i_N+1}) &= A_{i,i_p+1} (\phi^{i,i_{p+1}} (A_{i_{p+2},i_{p+1}+1} \dots A_{i_{N+1},i_N+1}))
\end{aligned}
$$

for $j = i_p+2, \dots, i_{p+1}$ and $i = i_p+1, \dots, i_{p+1}$, so that vanishing of the telescoping sum implies that $\sum_{N=0}^{2n} B_N = 0$. □

Lemma 5.2 *Suppose that* $\{v_p \mid p = 0, \dots, 2n\}$ *is a basis of* $\mathrm{Ker}\,(\mathrm{ad}\, f_{odd})$ *such that* $\Delta_{J_{\bar{v}_p}} = \frac{1}{2}(2n + 1 - p)$. *Take* $V_p \in W(\overline{\mathfrak{g}}, f_{prin}, k)$ *of the form* $V_p = J_{\bar{v}_p} + w_p$ *satisfying the conditions*

(i) V_p *and* w_p *have the conformal weight* $\frac{1}{2}(2n + 1 - p)$,
(ii) w_p *lies in the differential algebra generated by* $J_{\bar{a}}$ *for* $\Delta_{J_{\bar{a}}} < \Delta_{V_p}$.

Then, the set $\{V_p \mid p = 0, \ldots, 2n\}$ *freely generates the* W*-algebra* $W(\bar{\mathfrak{g}}, f_{prin}, k)$.

Proof A generating set of the form $\{V'_p = J_{\bar{v}_p} + w'_p \mid p = 0, \ldots, 2n\}$ satisfying the required conditions (i) and (ii) exists by Theorem 4.11. Set

$$\mathcal{W}_m := \text{subalgebra freely generated by } \{V_m, V_{m+1}, \ldots, V_{2n}\},$$
$$\mathcal{W}'_m := \text{subalgebra freely generated by} \{V'_m, V'_{m+1}, \ldots, V'_{2n}\}.$$

We will show by a (reverse) induction that $\mathcal{W}_m = \mathcal{W}'_m$ for all $m = 0, \ldots, 2n$. Note that $\mathcal{W}_{2n} = \mathcal{W}'_{2n}$, since w_{2n} and w'_{2n} are constants. Now suppose that $\mathcal{W}_p = \mathcal{W}'_p$ for some $p \leq 2n$. Then, $V_{p-1} - V'_{p-1} \in \mathcal{W}_p = \mathcal{W}'_p$ by condition (ii). Hence, we can conclude that $V'_{p-1} = V_{p-1} + (w'_p - w_p) \in \mathcal{W}_{p-1}$ and, similarly, $V_{p-1} \in \mathcal{W}'_{p-1}$. This shows that $\mathcal{W}_{p-1} = \mathcal{W}'_{p-1}$. Thus, $\mathcal{W}'_0 = \mathcal{W}_0$ and since $W(\bar{\mathfrak{g}}, f_{\text{prin}}, k) = \mathcal{W}'_0$, the lemma follows. □

Theorem 5.3 *The set of coefficients* $\{W_p \mid p = 0, \ldots, 2n\}$ *of* cdet $\mathcal{A}$ *freely generates* $W(\bar{\mathfrak{g}}, f_{prin}, k)$ *as a differential algebra.*

Proof Note that for $i \geq j$ we have

$$\Delta_{A_{i,j}(X)} = \frac{1}{2}(i - j + 1) + \Delta_X,$$

and each term in (5.1) satisfies

$$\Delta_{A_{i_1,i_0+1} A_{i_2,i_1+1} \ldots A_{i_{N+1},i_N+1}(X)} = \frac{2n+1}{2} + \Delta_X.$$

A direct calculation gives

$$W_{2n-k} = \sum_{l=1}^{2n+1-k} (-1)^{kl} J_{k+l,l} + w_{2n-k} \qquad \text{for} \quad k = 0, 1, \ldots, 2n,$$

where $\Delta_{2n-k} = \frac{2n+1}{2} - \frac{2n-k}{2}$ and w_{2n-k} can be expressed as a normally ordered product of the elements $J_{i,j}$ with $0 \leq i - j \leq k$ and their derivatives. It remains to apply Lemma 5.2. □

Example 5.4 Let $\mathfrak{g} = \mathfrak{gl}(2|1)$. Then, $f_{\text{prin}} = E_{21} + E_{32}$ and

$$\mathcal{A} = \begin{bmatrix} A_{1,1} & -1 & 0 \\ A_{2,1} & A_{2,2} & -1 \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix}.$$

The column determinant of $\mathcal{A}$ is

$$\begin{aligned} \text{cdet}\, \mathcal{A} &= A_{1,1}A_{2,2}A_{3,3} + A_{3,1} + A_{2,1}A_{3,3} + A_{1,1}A_{3,2} \\ &= (kS)^3 + W_2 S^2 + W_1 S + W_0. \end{aligned}$$

where

$$\begin{aligned} W_2 &= k^2(J_{1,1} + J_{2,2} + J_{3,3}),\\ W_1 &= k(-J_{1,1}J_{2,2} - J_{1,1}J_{3,3} - J_{2,2}J_{3,3} - J_{2,1} + J_{3,2} - kJ'_{2,2}),\\ W_0 &= -J_{1,1}J_{2,2}J_{3,3} - J_{2,1}J_{3,3} + J_{1,1}J_{3,2} + J_{3,1}\\ &\quad + kJ'_{3,2} + kJ_{1,1}J'_{3,3} - kJ'_{2,2}J_{3,3} + kJ_{2,2}J'_{3,3} + k^2J''_{3,3}, \end{aligned}$$

and $X' := [S, X]$. Hence, $W(\overline{\mathfrak{g}}, f_{\text{prin}}, k)$ is freely generated by W_0, W_1 and W_2. □

As in [3], by taking the quotient of the W-algebra $W(\overline{\mathfrak{g}}, f_{\text{prin}}, k)$ over the supersymmetric vertex algebra ideal generated by the elements $J_{i,j}$ with $i > j$ we recover the presentation of the W-algebra via the *Miura transformation*; cf. [10,17,18]:

$$\operatorname{cdet} \mathcal{A} \mapsto \big(kS + J_{1,1}\big)\big(kS - J_{2,2}\big)\big(kS + J_{3,3}\big)\dots\big(kS - J_{2n,2n}\big)\big(kS + J_{2n+1,2n+1}\big).$$

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