Review

A : $\mathbb{Z}[q^{\pm 1/2}]$-algebra in a skew field K.

$\mathcal{S} = (\{x_i\}_{i\in J},\ L = (\lambda_{ij})_{i,j\in J},\ \tilde{B} = (b_{ij})_{i\in J,\, j\in J_{ex}})$: Quantum seed in A if

(i) $(L, \tilde{B})$: compatible pair

(ii) $\{x_i\}_{i\in I}$: algebraically independent L-commuting family

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Main Question

For a given $k \in J_{ex}$,

$\mu_k(\mathcal{S}) := (\{\mu_k(x)_i\}_{i\in J},\ \mu_k(L),\ \mu_k(\tilde{B}))$: Quantum seed in K?

$$\mu_k(x)_i := \begin{cases} x^{a'} + x^{a''} & \text{if } i = k \\ x_i & \text{if } i \neq k \end{cases}$$

where $a'_t = \begin{cases} -1 & \text{if } t = k \\ \max(0, b_{tk}) & \text{if } t \neq k \end{cases}$ $\quad a''_t = \begin{cases} -1 & \text{if } t = k \\ \max(0, -b_{tk}) & \text{if } t \neq k \end{cases}$

(Compatibility of $(\mu_k(L), \mu_k(\tilde{B}))$: proved last week)

① $\{\mu_k(x)_i\}_{i\in J}$: $\mu_k(L)$-commuting family in K

$\iff \quad \mu_k(x)_i\, \mu_k(x)_j = q^{(\mu_k(L))_{ij}}\, \mu_k(x)_j\, \mu_k(x)_i \in K$

② $\{\mu_k(x)_i\}_{i\in J}$: algebraically independent.

$\iff \quad P(\mu_k(L)) \longmapsto K$ is injective.

$\qquad\qquad X_i \longmapsto x_i$

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① Case 1 $\quad i = k,\ j = k$

$(\mu_k(L))_{kk} = 0 \implies \mu_k(x)_k\, \mu_k(x)_k = q^{(\mu_k(L))_{kk}}\, \mu_k(x)_k\, \mu_k(x)_k$

Case 2 $\quad i \neq k,\ j = k$

$$(\mu_k(L))_{ik} = -\lambda_{ik} + \sum_{t\in J} \max(0, -b_{tk})\,\lambda_{it}$$

$$= \lambda_{ki} + \sum_{t\in J} \max(0, -b_{tk})\,\lambda_{it}$$

Lemma If $i \neq k$, $j = k$, then

(L,B) Compatible 쓰인 곳 !!!

$$\sum_{t \in J} \max(0, -b_{tk}) \lambda_{it} = \sum_{t \in J} \max(0, b_{tk}) \lambda_{it}$$

(pf) $\sum_{\alpha \in J} \lambda_{i\alpha} b_{\alpha k} = 0$. Let $J_1 = \{\alpha \in J \mid b_{\alpha k} > 0\}$, $J_2 = \{\alpha \in J \mid b_{\alpha k} < 0\}$.

$$\sum_{t \in J} \max(0, -b_{tk}) \lambda_{it} = -\sum_{t \in J_2} \lambda_{it} b_{tk} = \sum_{t \in J_1} \lambda_{it} b_{tk}$$

$$\sum_{t \in J} \max(0, b_{tk}) \lambda_{it} = \sum_{t \in J_1} \lambda_{it} b_{tk}$$

$\square$

Recall

$$X^{a} X^{b} = q^{\frac{1}{2} \sum_{i,j \in J} a_i b_j \lambda_{ij}} X^{a+b}$$

$$= q^{-\frac{1}{2} \sum_{i,j \in J} b_i a_j \lambda_{ij}} X^{b+a} = q^{-\sum_{i,j \in J} b_i a_j \lambda_{ij}} X^{b} X^{a}$$

$$(\mu_k(x))_i \, (\mu_k(x))_k = x_i \left(x^{a'} + x^{a''}\right)$$

$$= q^{-\sum_{t \in J} a'_t \lambda_{ti}} x^{a'} x_i + q^{-\sum_{t \in J} a''_t \lambda_{ti}} x^{a''} x_i$$

$$= q^{\lambda_{ki} - \sum_{t \neq k} \max(0, b_{tk}) \lambda_{ti}} x^{a'} x_i$$

$$+ \, q^{\lambda_{ki} - \sum_{t \neq k} \max(0, -b_{tk}) \lambda_{ti}} x^{a''} x_i$$

$$\overset{\text{Lemma}}{=} q^{\lambda_{ki} + \sum_{t \in J} \max(0, -b_{tk}) \lambda_{it}} \left(x^{a'} + x^{a''}\right) x_i$$

$$= q^{(\mu_k(L))_{ik}} (\mu_k(x))_k \, (\mu_k(x))_i$$

Case 3 $i = k$, $j \neq k$ Similar to **Case 2**

Case 4 $i \neq k$, $j \neq k$

$$(\mu_k(x))_i \, (\mu_k(x))_j = x_i x_j = q^{\lambda_{ij}} x_j x_i = q^{(\mu_k(L))_{ij}} (\mu_k(x))_j \, (\mu_k(x))_i$$

$\square$

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$$\begin{tikzcd}
\langle X_i \rangle_{i\in I} = P(L) \arrow[r, hook] \arrow[rr, bend left, "\varphi_1"] & A \arrow[r, hook] & K \\
\langle X_i, \mu_k(X)_k \mid i \neq k \rangle = P(\mu_k(L)) \arrow[urr, "\varphi_2"'] & &
\end{tikzcd}$$

<u>We have</u> • $\varphi_1 : X_i \longmapsto x_i$ is an algebra monomorphism

• $\varphi_2 : \left\{ \begin{array}{l} X_i \longmapsto x_i \quad (i \neq k) \\ \mu_k(X_k) \longmapsto \mu_k(x)_k \end{array} \right\}$

<u>We want to show</u> φ_2 is an algebra monomorphism.

Since

$$\mu_k(X)_i \, \mu_k(X)_j = q^{(\mu_k(L))_{ij}} \mu_k(X)_j \, \mu_k(X)_i$$

$$\mu_k(x)_i \, \mu_k(x)_j = q^{(\mu_k(L))_{ij}} \mu_k(x)_j \, \mu_k(x)_i \qquad \forall i,j \in J,$$

φ_2 is an algebra homomorphism.

<u>Universal property of skew field of fractions.</u>

P : non-commutative ring with unity.

A skew field $\mathcal{F}$ containing P is called <u>a universal skew field of fractions of P</u> if

(i) there is no proper subskewfield of $\mathcal{F}$ containing P

(ii) for any skew field K and any morphism of rings $\varphi : P \to K$, $\exists$ a subring $\mathcal{F}_0$ of $\mathcal{F}$ with $P \subseteq \mathcal{F}_0$ and a morphism of rings $\overline{\varphi} : \mathcal{F}_0 \longrightarrow K$ extending φ with the property

$$\forall x \in \mathcal{F}_0 , \quad x^{-1} \in \mathcal{F}_0 \iff \overline{\varphi}(x) \neq 0$$

Lemma P : non-commutative ring with unity.

$\mathcal{F}$: universal skewfield of fractions of P

For a skewfield K, if a morphism of rings $\varphi : P \to K$ is injective, then φ extends to $\overline{\varphi} : \mathcal{F} \to K$ and injective.

(pf) Use the universal property (ii).

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\begin{tikzcd}
\langle X_i \rangle_{i \in I} = P(L) \arrow[dr] \arrow[drr, bend left, "\varphi_1"] & & \\
& \mathcal{F}(L) \arrow[r, "\overline{\varphi_1}"'] \arrow[dr, phantom, "\circlearrowright"] & K \\
\langle X_i, \mu_k(X)_k \mid i \neq k \rangle = P(\mu_k(L)) \arrow[ur] \arrow[urr, "\varphi_2"'] & &
\end{tikzcd}
$$

$\therefore$ φ_2 is injective.

$\square$